Lecture 6: Integrality of slopes of ghost series at classical weights

Goal Compute $NP(G^{(\varepsilon)}(w_k,-))$ for $k \equiv k_\varepsilon \bmod p-1$.

Notation Suppose we have a list of pts $P_i = (i, A_i)$, $A_i \in \mathbb{R}$,

$i = m, m+1, \dots, n$.

We shift them down relative to a linear function $y = ax+b$ $(a, b \in \mathbb{R})$

by transforming them into $Q_i = (i, A_i - ai - b)$, $i = m, m+1, \dots, n$.

Facts (1) $\forall i \neq j$ slope of $\overline{Q_iQ_j}$ = slope of $\overline{P_iP_j} - a$.

(2) if the convex hull of P_i's is a straight line

then the same is true for Q_i's.

(3) Fix i_0. If P_{i_0} is a vertex of the convex hull of P_i's,

then Q_{i_0} is a vertex of the convex hull of Q_i's.

- $\Delta'_{k,l} = v_p\left(g_{\frac{1}{2}d_k^{Iw}+l,\hat{k}}(w_k)\right) - \frac{k-2}{2}\cdot l$
- Δ_k = Convex hull of $(l, \Delta'_{k,l})$ with $|l| \le \frac{1}{2}d_k^{new}$ $\longleftrightarrow$ Convex hull of $(n, v_p(g_n(w_\star)))$.

$n \in \overline{nS}_{w_\star,k} \subseteq \left[\frac{1}{2}d_k^{Iw} - \frac{1}{2}d_k^{new}, \frac{1}{2}d_k^{Iw} + \frac{1}{2}d_k^{new}\right]$.

Here $\overline{nS}_{w_\star,k} = \left(\frac{1}{2}d_k^{Iw} - L_{w_\star,k}, \frac{1}{2}d_k^{Iw} + L_{w_\star,k}\right)$.

$L_{w_\star,k}$ is the largest integer in $\{1, \dots, \frac{1}{2}d_k^{new}\}$

s.t. $v_p(w_\star - w_k) \ge \Delta_{k,L_{w_\star,k}} - \Delta_{k,L_{w_\star,k}-1}$.

Goal Show that the convex hull of the pts $(n, v_p(g_n(w_\star)))$ for $n \in \overline{nS}_{w_\star,k}$

is a straight line and we compute its slope.

Note $v_p(g_n(w_\star)) = v_p(g_{n,\hat{k}}(w_\star)) + m_n(k)\cdot v_p(w_\star - w_k)$, $g_n(w) = \prod (w - w_{k'})^{m_n(k')}$.

<u>Lemma</u> The convex hull of the pts

$$(n,\ v_p(g_{n,\tilde{k}}(w_k)) + m_n(k)\cdot v_p(w_* - w_k)),\quad n\in \overline{nS_{w_*,k}}$$

is a stright line of slope $\frac{k-2}{2}$.

<u>Proof</u> Write $n = \frac{1}{2}d_k^{I_w} + l \Rightarrow m_n(k) = \frac{1}{2}d_k^{new} - |l|$

$\rightsquigarrow$ the above pts can be written as

$$P_l = \left(\tfrac{1}{2}d_k^{I_w} + l,\ \Delta'_{k,l} + \tfrac{k-2}{2}l + \left(\tfrac{1}{2}d_k^{new} - |l|\right)\cdot v_p(w_* - w_k)\right)$$

for $l\in[-L,L]$, $L = L_{w_*,k}$ for simplicity.

We shift P_l's relative to the linear function

$$y = \tfrac{k-2}{2}\left(x - \tfrac{1}{2}d_k^{I_w}\right) + \Delta_{k,L} - \left(\tfrac{1}{2}d_k^{I_w} - L\right) v_p(w_* - w_k)$$

and we get

$$Q_l = \left(\tfrac{1}{2}d_k^{I_w} + l,\ \Delta'_{k,l} - \Delta_{k,L} + (L - |l|)\cdot v_p(w_* - w_k)\right).$$

By def'n $nS_{w_*,k} \Rightarrow v_p(w_* - w_k) \geq \Delta_{k,L} - \Delta_{k,L-1}$

$$\geq \Delta_{k,L-1} - \Delta_{k,L-2} \geq \cdots \geq \Delta_{k,|l|+1} - \Delta_{k,|l|}.$$

$$(L - |l|)\, v_p(w_* - w_k) \geq \Delta_{k,L} - \Delta_{k,|l|} \geq \Delta_{k,L} - \Delta'_{k,l}.$$

$\Rightarrow$ y coordinate of $Q_l \geq 0$,

"=" holds when $l = \pm L$.

$\Rightarrow$ convex hull of Q_l's is a straight line of slope 0. □

<u>Lemma</u> Let $k = k_\varepsilon + (p-1)k_0$, $l = 1, \dots, \frac{1}{2}d_k^{new}$.

Let $k' = k_\varepsilon + (p-1)k'_0$ be another wt

s.t. one of the following conditions holt:

(1) either $d_{k'}^{ur}$ or $d_{k'}^{I_w} - d_{k'}^{ur} \in \left(\frac{1}{2}d_k^{I_w} - l,\ \frac{1}{2}d_k^{I_w} + l\right)$

(2) $\frac{1}{2}d_{k'}^{I_w} \in \left[\frac{1}{2}d_k^{I_w} - l,\ \frac{1}{2}d_k^{I_w} + l\right]$.

Then $\Delta_{k,l} - \Delta'_{k,l-1} - v_p(w_k - w_{k'}) \geq \frac{1}{2}(2l-1)$

$\left(\Delta'_{k,l} - \Delta'_{k,l-1} \geq \frac{3}{2} + \frac{p-1}{2}(l-1). \right)$

Prop Let $nS_{w_\star,k} = \left(\frac{1}{2}d_k^{\mathrm{Iw}} - L_{w_\star,k},\ \frac{1}{2}d_k^{\mathrm{Iw}} + L_{w_\star,k}\right)$ be a near-Steinberg range.

(1) $\forall k' = k_\varepsilon + (p-1)k'_\bullet$. $v_p(w_{k'} - w_k) \geq \Delta_{k,L_{w_\star,k}} - \Delta_{k,L_{w_\star,k}-1}$

then $\frac{1}{2}d_{k'}^{\mathrm{Iw}} \notin \overline{nS_{w_\star,k}}$. $d_{k'}^{\mathrm{ur}},\ d_{k'}^{\mathrm{Iw}} - d_{k'}^{\mathrm{ur}} \notin nS_{w_\star,k}$.

(2) For k' in (1), $m_n(k')$ is linear in n when $n \in \overline{nS_{w_\star,k}}$.

(3) The following two lists of pts

$$P_n = (n, v_p(g_{n,\hat{k}}(w_\star))), \quad Q_n = (n, v_p(g_{n,\hat{k}}(w_k))), \quad n \in \overline{nS_{w_\star,k}}$$

are shifts of each other relative to a linear function with slope in $\mathbb{Z} + \mathbb{Z}\alpha$, $\alpha = \max\{v_p(w_\star - w_{k'}) : w_{k'}$ is a zero of $g_n(w)$, $n \in nS_{w_\star,k}\}$.

(4) More generally, let $\underline{k} = \{k_1, \cdots, k_r\}$, $k_i \equiv k_\varepsilon \bmod p-1$.

Suppose $\exists$ an interval $[n_-, n_+]$ s.t. $\forall k' \notin \underline{k}$

with $v_p(w_{k'} - w_k) \geq v_p(w_\star - w_k)$

then the ghost multiplicity $m_n(k')$ is linear in n for $n \in [n_-, n_+]$.

Then the lists of pts $P_n = (n, A_n, v_p(g_{n,\hat{\underline{k}}}(w_\star)))$

$$Q_n = (n, A_n, v_p(g_{n,\hat{\underline{k}}}(w_k))) \qquad (A_n \in \mathbb{R})$$

are shifts of each other by a linear function w/ slope in $\mathbb{Z} + \mathbb{Z}\beta$. $\beta = \max\{v_p(w_\star - w_{k'}) \mid w_{k'}$ is a zero of $g_{n,\hat{\underline{k}}}(w)$, $n \in [n_-, n_+]\}$.

Proof (1) is an immediate conseq'ce of the previous lemma.

(2) follows from the def of $\{m_n(k')\}_{n \geq 1}$:

$$\underbrace{0, \cdots,}_{d_{k'}^{\mathrm{ur}}} \Big| \underbrace{0, 1, 2, \cdots, \tfrac{1}{2}d_{k'}^{\mathrm{new}}}_{\longleftarrow \text{ linear } \longrightarrow}, \Big| \underbrace{\tfrac{1}{2}d_{k'}^{\mathrm{new}} - 1, \cdots, 2, 1, 0}_{\longleftarrow \text{ linear } \longrightarrow} \Big| \underbrace{0, \cdots}_{\longleftarrow \text{ linear } \longrightarrow} \Big|$$

(3) $v_p(g_{n,\hat{k}}(w_{k^*})) - v_p(g_{n,\hat{k}}(w_k))$

$$= \sum_{\substack{k' \equiv k \, (p-1) \\ k' \neq k}} m_n(k') \left(v_p(w_{k^*} - w_{k'}) - v_p(w_k - w_{k'}) \right).$$

$\boxed{\in \mathbb{Z} + \mathbb{Z}\alpha}$

If $v_p(w_{k'} - w_k) < \Delta_{k,L} - \Delta_{k,L-1}$ $(\leq v_p(w_{k^*} - w_k))$.

$\Rightarrow v_p(w_{k^*} - w_{k'}) = v_p(w_k - w_{k'})$.

If $v_p(w_{k'} - w_k) \geq \Delta_{k,L} - \Delta_{k,L-1}$ by (2).

$m_n(k')$ is linear for $n \in \overline{nS_{w_{k^*},k}}$. $\quad (L = L_{w_{k^*},k}.)$ $\square$

Rmk The previous lemma gives

$$\Delta_{k,l} - \Delta'_{k,l-1} - v_p(w_{k'} - w_k) \geq \frac{1}{2}(2l-1)$$

if we have $w_{k'}$ very close to w_k,

then the exclusions in (1) most holds.

Cor Fix $\overline{nS_{w_{k^*},k}}$ as before. $w_{k^*} \neq w_k$.

(1) Assume that w_{k^*} is not a zero of $g_n(w)$ for any (or some) $n \in \overline{nS_{w_{k^*},k}}$.

Then the lower convex hull of pts $(n, v_p(g_n(w_{k^*})))$ for $n \in \overline{nS_{w_{k^*},k}}$ is a straight line.

Moreover, the slope of this straight line belongs to $\frac{a}{2} + \mathbb{Z} + \mathbb{Z}\alpha$. $(*)$

(2) Assume that $w_{k^*} = w_{k_1}$ is a zero of $g_n(w)$ for $\overline{\text{any}}$ (or some) $n \in \overline{nS_{w_{k^*},k}}$.

Then the lower hull of pts $(n, v_p(g_{n,\hat{k}}(w_{k_1})))$, $n \in \overline{nS_{w_{k^*},k}}$ is a straight line w/ slope $\in \frac{a}{2} + \mathbb{Z}$. b/c $m_n(k_1)$ linear in n.

Proof (1) For $n \in \overline{nS_{w_{k^*},k}}$.

$$(n, v_p(g_n(w_{k^*}))) \xleftrightarrow[\text{with slope} \in \mathbb{Z} + \mathbb{Z}\alpha]{\text{shift rel to a lin func}} (n, v_p(g_{n,\hat{k}}(w_k) + m_n(k) v_p(w_{k^*} - w_k)).$$

$\parallel$

$(n, v_p(g_{n,\hat{k}}(w_k)) + m_n(k) v_p(w_{k^*} - w_k))$

lower convex hull = stright line of slope $\frac{k-2}{2}$.

note $k \equiv k_\varepsilon \mod p-1 \equiv 2 + \{a + 2S_\varepsilon\} \equiv 2 + a + 2S_\varepsilon \mod p-1$.

$\frac{k-2}{2} \in \frac{a}{2} + \mathbb{Z}$.

Cor For any $k \equiv k_\varepsilon \mod p-1$,

(1) The slopes of $NP(G^{(\varepsilon)}(w_k, -))$ with multi 1 are all integers

(2) Other slopes of $NP(G^{(\varepsilon)}(w_k, -))$ always have even multiplicities and belong to $\frac{a}{2} + \mathbb{Z}$.

For (1): $v_p(g_n(w_k)) \in \mathbb{Z}$.

For (2): we need to compute the slope of lower convex hull of $NP(G^{(\varepsilon)}(w_k, -))$ on a max'l $nS_{w_k, k'}$.

This follows from the previous corollary (1).

Prop Fix $k_0 \equiv k_\varepsilon \mod p-1$. $l \in \{1, \dots, \frac{1}{2} d_{k_0}^{new} - 1\}$. TFAE:

(1) $(l, \Delta'_{k_0, l})$ is not a vertex of Δ_{k_0}.

(2) $(w_{k_0}, \frac{1}{2} d_{k_0}^{Iw} + l, k_1)$ is near-Steinberg for some $k_1 > k_0$

i.e. $\frac{1}{2} d_{k_0}^{Iw} + l \in nS_{w_{k_0}, k_1}$.

(3) $(w_{k_0}, \frac{1}{2} d_{k_0}^{Iw} - l, k_2)$ is near-Steinberg for some $k_2 < k_0$.