Lecture 7: Irreducible components of eigencurves

<u>Goal</u> Ghost $\Rightarrow$ finiteness of irred comps of eigencurves.

<u>Main Thm</u> $\widetilde{H}$: $\mathcal{O}[\![K_p]\!]$-proj arith mods of type $\bar{r}_p$ $(\bar{\rho})$
with multiplicity $m(\widetilde{H})$

ε : relevant char

$C_{\widetilde{H}}^{(\varepsilon)}(w,t)$: char power series of U_p on
abstract p-adic forms assoc to $\widetilde{H}$.

$G_{\bar{\rho}}^{(\varepsilon)}(w,t)$: ghost series (depends only on $\bar{\rho}$). $\quad (\bar{r}_p|_{I_{\mathbb{Q}_p}} \simeq \bar{\rho}^{ss})$.

<u>Then</u> for any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, $NP(G_{\widetilde{H}}^{(\varepsilon)}(w_\star,-))$ is the same as
$NP(G_{\bar{\rho}}^{(\varepsilon)}(w_\star,-))$ stretched in both x-, y-directions by $m(\widetilde{H})$
except the slope-zero part.

- has length $m'(\widetilde{H})$ when $\bar{r}_p$ is split, $\varepsilon = \omega^b \times \omega^{a+b}$.
- has length $m''(\widetilde{H})$ when $\bar{r}_p$ is non-split. $\varepsilon = \omega^{a+b+1} \times \omega^{b-1}$.

(<u>Note</u> $\bar{r}_p$ split $\rightsquigarrow$ $\widetilde{H} \simeq (\mathrm{Proj}_{\mathcal{O}[\![K_p]\!]}\sigma_{a,b})^{\oplus m'(\widetilde{H})} \oplus (\mathrm{Proj}_{\mathcal{O}[\![K_p]\!]}\sigma_{b,a})^{\oplus m''(\widetilde{H})}$.)

<u>Def'n</u> Fix $\lambda \in (0,1)$. Take $\mathcal{W}_{\geq\lambda} := \mathrm{Sp}\, E\langle w/p^\lambda\rangle$.

(1) A Fredholm series of $\mathcal{W}_{\geq\lambda}$ is $F(w,t) \in E\langle w/p^\lambda\rangle[\![t]\!]$.
s.t. $F(w,0)=1$ and $F(w,t)$ converges on $\underbrace{\mathcal{W}_{\geq\lambda}\times(\mathbb{A}^1)^{rig}}$.
$\qquad \cup\!\!\!\mid$
$\mathcal{Z}(F)$ = zero locus of F.

(2) A Fredholm series $F(w,t)$ is called of ghost type $(\bar{r}_p, \varepsilon)$
if $\forall w_\star \in \mathcal{W}_{\geq\lambda}(\mathbb{C}_p)$, $NP(F(w_\star,-))$ is the same as $NP(G_{\bar{\rho},\mathrm{nord}}^{(\varepsilon)}(w_\star,-))$
stretched in both x-, y-directions by $m(F) \in \mathbb{N}$.
$\qquad\qquad \uparrow$
$\qquad\qquad$ multiplicity

Lemma $C^{(\varepsilon)}_{\tilde{H}} = C^{(\varepsilon)}_{\tilde{H},\mathrm{ord}} \cdot C^{(\varepsilon)}_{\tilde{H},\mathrm{nord}}$,

where $C^{(\varepsilon)}_{\tilde{H},\mathrm{nord}}$ is of ghost type $(\bar{r}_p, \varepsilon)$ w/ multi $m(\tilde{H})$.

Proof Main thm + Weierstrass preparation

$\Rightarrow C^{(\varepsilon)}_{\tilde{H}}(w,t) = C^{(\varepsilon)}_{\tilde{H},\mathrm{ord}} \cdot C^{(\varepsilon)}_{\tilde{H},\mathrm{nord}}$,

- $C^{(\varepsilon)}_{\tilde{H},\mathrm{nord}}(w,t)$ is of ghost type $\tilde{m}$, with multiplicity $m(\tilde{H})$
- $C^{(\varepsilon)}_{\tilde{H},\mathrm{ord}}$ is a polynomial of deg $m'(\tilde{H})$ or $m''(\tilde{H})$.

Key Technical lemma

$\check{\mathcal{O}}$:= completion of max unram ext'n of $\mathcal{O}$, $\check{E}$:= Frac $\check{\mathcal{O}}$.

$r \in \mathbb{Q}_{>0}$, $D(w_*, r) = \{ w \in \mathcal{W}_{\geq\lambda}(\mathbb{C}_p) : v_p(w - w_*) \leq r \}$

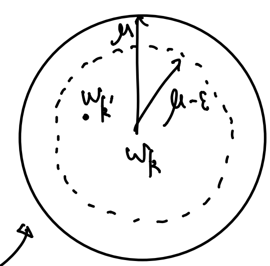
$\rightsquigarrow \eta_{w_*, r}$ assoc Gauss pt.

Slope derivatives: $\mu \in (\lambda, \infty) \cap \mathbb{Z}$.

$$\rightsquigarrow V^{+}_{w_*,\mu}(f) := \lim_{\varepsilon \to 0^+} \left(- \frac{\ln|f(\eta_{w_*,\mu-\varepsilon})| - \ln|f(\eta_{w_*,\mu})|}{(\ln p)\cdot \varepsilon} \right)$$

$$V^{\bar{\alpha}}_{w_*,\mu}(f) := \lim_{\varepsilon \to 0^+} \left(- \frac{\ln|f(\eta_{w_*+\alpha p^{\mu},\mu+\varepsilon})| - \ln|f(\eta_{w_*+\alpha p^{\mu},\mu})|}{(\ln p)\cdot \varepsilon} \right)$$

↑ depends only on $\bar{\alpha}$.



Then $V^{+}_{w_*,\mu}(f) + \sum_{\bar{\alpha} \in \bar{\mathbb{F}}} V^{\bar{\alpha}}_{w_*,\mu}(f) = 0$

($V^{\bar{\alpha}}_{w_*,\mu}(f) = 0$ for almost all $\bar{\alpha}$.)

E.g. $g_n(w)$ "ghost polynomial". $g_n(w) = \prod (w - w_k)^{m_n(k)}$.

$$\Rightarrow V^{+}_{w_*,\mu}(g_n) = \sum_{v_p(w_{k'} - w_k) > \mu} m_n(k') - \frac{|w - w_{k'}|_{\eta_{w_k,\mu-\varepsilon}} - |w - w_{k'}|_{\eta_{w_k,\mu}}}{\ln p \cdot \varepsilon} = m_n(k'))$$

Thm $F(w,t)$ Fredholm series of ghost type with multi $m(F)$.

& If Fredholm series $H(w,t) \mid F(w,t)$, then H is of ghost type with multi $m(H) \leq m(F)$.

Cor $Z(C^{(\varepsilon)}_{\tilde{H},\mathrm{nord}})$ has only fin many irred components $\le m(\tilde{H})$.

<u>Proof of Thm</u> $F(w,t) = H(w,t)\cdot H'(w,t)$,

$w_* \in \mathcal{W}_{\ge\lambda}(\mathbb{C}_p)$ s.t. $(n, v_p(g_n(w_*)))$ is a vertex of ghost NP.

Form an open subspace of $\mathcal{W}_{\ge\lambda}$:

$$\underset{\text{connected}}{\mathrm{Vtx}_{n,\ge\lambda}} := \mathcal{W}_{\ge\lambda} \setminus \bigcup_{\substack{k\equiv k_\varepsilon (p-1)\\ n\in(d_k^{ur},\, d_k^{Iw}-d_k^{ur})}} D\left(w_k,\ \Delta_{k,|\frac{1}{2}d_k^{Iw}-n|+1} - \Delta_{k,|\frac{1}{2}d_k^{Iw}-n|}\right)$$

⇝ assoc Berkovich space $\mathrm{Vtx}^{\mathrm{Berk}}_{n,\ge\lambda}$.

<u>Step I</u> The total multiplicity of n smallest slopes of ghost NP in H is constructed for $w_* \in \mathrm{Vtx}^{\mathrm{Berk}}_{n,\ge\lambda}$, denoted by $m(H,n)$ ($\overset{?}{=} m(H)\cdot n$).

<u>Step II</u> It is known $\exists! k \equiv k_\varepsilon$ s.t. $\frac{1}{2}d_k^{Iw} = n-1$.

<u>Claim</u> $\forall \varepsilon \in (0,\frac{1}{2})$, $\forall \alpha \in \mathcal{O}_{\mathbb{C}_p}$,

(1) $\eta_{w_k,\, \Delta_{k,1}-\Delta_{k,0}-\varepsilon}$ belongs to $\mathrm{Vtx}^{\mathrm{Berk}}_{n,\ge\lambda}$, $\mathrm{Vtx}^{\mathrm{Berk}}_{n-1,\ge\lambda}$

(2) $\eta_{w_k+\alpha p^{\Delta_{k,1}-\Delta_{k,0}},\, \Delta_{k,1}-\Delta_{k,0}+\varepsilon}$ belongs to $\mathrm{Vtx}^{\mathrm{Berk}}_{n,\ge\lambda}$, $\mathrm{Vtx}^{\mathrm{Berk}}_{n-2,\ge\lambda}$, but not $\mathrm{Vtx}^{\mathrm{Berk}}_{n-1,\ge\lambda}$.

<u>Step III</u> Granting step I, II, we conclude the proof.

$m(H) := m(H,1)$. Will prove inductively that $m(H,n) = n\cdot m(H)$.

- $n=1$: ok.
- Assume the statement holds for smaller n.

Take k as in Step II. $H(w,t) = \sum_{m\ge0} h_m(w)\cdot t^m$.

<u>Step II (1)</u> $\Rightarrow$ $|h_{m(H,n)}(\eta_{w_k,\Delta_{k,1}-\Delta_{k,0}-\varepsilon})|$

$$= |g_{n-1}^{m(H)}(\eta_{w_k,\Delta_{k,1}-\Delta_{k,0}-\varepsilon})\cdot (g_n/g_{n-1})^{m(H,n)-m(H,n-1)}(\eta_{w_k,\Delta_{k,1}-\Delta_{k,0}-\varepsilon})|.$$

By continuity this holds for $\varepsilon=0$ as well.

$$\Rightarrow V^{+}_{w_k,\Delta_{k,1}-\Delta_{k,0}}(h_{m(H,n)}) = V^{+}_{w_k,\Delta_{k,1}-\Delta_{k,0}}\left(g_{n-1}^{m(H)}\cdot (g_n/g_{n-1})^{m(H,n)-m(H,n-1)}\right)$$

Step II (2) $\Rightarrow$ $\eta_{w_k + \alpha p^{\Delta_{k,1}-\Delta_{k,0}},\ \Delta_{k,1}-\Delta_{k,0}+\varepsilon}$

ghost NP at these pts is a straight line from $n-2$ to n

$$\Rightarrow V^{\bar{\alpha}}_{w_k,\Delta_{k,1}-\Delta_{k,0}}(h_{m(H,n)}) = V^{\bar{\alpha}}_{w_k,\Delta_{k,1}-\Delta_{k,0}}\left(g_{n-2}^{m(H)}\cdot(g_n/g_{n-2})^{\frac{m(H,n)-m(H,n-2)}{2}}\right).$$

Sum up

$$0 = V^{+}_{w_k,\Delta_{k,1}-\Delta_{k,0}}\Big(\underbrace{g_{n-1}^{m(H)}\cdot(g_n/g_{n-1})^{m(H,n)-m(H,n-1)}}_{A}\Big)$$

$$+ V^{\bar{\alpha}}_{w_k,\Delta_{k,1}-\Delta_{k,0}}\Big(\underbrace{g_{n-2}^{m(H)}\cdot(g_n/g_{n-2})^{\frac{m(H,n)-m(H,n-2)}{2}}}_{B}\Big).$$

$$= V^{+}_{w_k,\Delta_{k,1}-\Delta_{k,0}}(A/B).$$

where $A/B = \left(\frac{g_n\cdot g_{n-2}}{g_{n-1}^2}\right)^{\frac{m(H,n)-m(H,n-1)-m(H)}{2}}$

(Use $m(H,n-1) = m(H,n-2)+m(H)$.)

To show that $m(H,n)-m(H,n-1)-m(H)=0$,

it suffices to show $V^{+}_{w_k,\Delta_{k,1}-\Delta_{k,0}}\left(\frac{g_n g_{n-2}}{g_{n-1}^2}\right) \neq 0$

$$\overset{\shortparallel}{V^+(g_n)+V^+(g_{n-2})-2V^+(g_{n-1})} \overset{\text{claim}}{=\!=} -2.$$

Pf of the claim

For $i\in\{n-2,n-1,n\}$,

$$V^{+}_{w_k,\Delta_{k,1}-\Delta_{k,0}}(g_i) = \sum_{\substack{v_p(w_{k'}-w_k)>\Delta_{k,1}-\Delta_{k,0}\\ k'\equiv k_\varepsilon}} m_i(k')$$

$m_{n-2}(k')+m_n(k')-2m_{n-1}(k')$

$m_i(k')$ is linear except for $i = d_{k'}^{ur},\ \frac{1}{2}d_{k'}^{Iw},\ d_{k'}^{Iw}-d_{k'}^{ur}$.

Recall $\frac{1}{2}d_k^{Iw} = n-1 \Rightarrow$ If $v_p(w_{k'}-w_k) > \Delta_{k,1}-\Delta_{k,0}$

then $\{n-2,n-1,n\}\subseteq nS_{w_{k'},k}$.

On the other hand, if $k'\neq k$ then $d_{k'}^{ur},\ \frac{1}{2}d_{k'}^{Iw},\ d_{k'}^{Iw}-d_{k'}^{ur}\notin nS_{w_{k'},k}$

$\Rightarrow$ $m_i(k')$ is linear for $i\in\{n-2,n-1,n\}$ except $k'=k$.

$m_{n-2}(k)+m_n(k)-2m_{n-1}(k) = -2.$