Lecture 8: On Paškūnas modules

§1 (φ,Γ)-modules and p-adic LLC for $GL_2(\mathbb{Q}_p)$

$\mathcal{O}_\mathcal{E} := \mathcal{O}[\![T]\!][\frac{1}{T}]^{\wedge}_p$, $\quad \mathcal{O}_\mathcal{E}/(p) = \mathbb{F}((T))$

$\varphi, T \underset{\mathcal{O}\text{-linear}}{\curvearrowright} \mathcal{O}_\mathcal{E}, \mathbb{F}((T))$, $\quad \varphi(T) = (1+T)^p - 1$

$\gamma \in \Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \xrightarrow[\chi]{\sim} \mathbb{Z}_p^\times$, $\quad \gamma(T) = (1+T)^{\chi(\gamma)} - 1$.

<u>Def'n</u> $? = \mathcal{O}_\mathcal{E}, \mathbb{F}((T))$ with $\mathcal{E} = \mathcal{O}_\mathcal{E}[\frac{1}{p}]$.

A (φ,Γ)-mod over $?$ is a finite free $?$-mod M equipped with commuting semilinear actions of φ.

- For $? = \mathcal{O}_\mathcal{E}, \mathbb{F}((T))$, M is called étale if $\underset{M \otimes_{?,\varphi} ?}{\overset{\|}{\varphi^* M}} \simeq M$.
- For $? = \mathcal{E}$, M is called étale if it is the base change of an étale (φ,Γ)-mod $/\mathcal{O}_\mathcal{E}$.

<u>Thm.</u> (Fontaine) $\exists$ rank-preserving equiv of cats

$$\left\{\begin{matrix}\text{étale } (\varphi,\Gamma)\text{-mods} \\ \text{over } \mathcal{E} \text{ or } \mathcal{O}_\mathcal{E} \text{ or } \mathbb{F}((T))\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} G_{\mathbb{Q}_p}\text{-rep'ns over} \\ E \text{ or } \mathcal{O} \text{ or } \mathbb{F}\end{matrix}\right\}$$

$$D \longmapsto V(D).$$

<u>Colmez's functor</u>

$P^+ := \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \curvearrowright \mathbb{F}\text{-v.s. } M.$

$\rightsquigarrow$ a structure of (φ,Γ)-mods over $\mathbb{F}((T))$

- $\mathbb{F}[\![\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1\end{pmatrix}]\!] \longrightarrow \mathbb{F}[\![T]\!]$, $\quad \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1\end{pmatrix} \simeq T,$

  $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} - 1 \longmapsto T \qquad \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} \rightsquigarrow \varphi.$

$\rightsquigarrow \varphi, \Gamma \circlearrowright M / \mathbb{F}[\![T]\!]$.

Def'n π sm adm finite length rep'n of $GL_2(\mathbb{Q}_p)$ over $\mathbb{F}$.

$$D(\pi) := \mathbb{F}((T)) \hat{\otimes}_{\mathbb{F}[\![T]\!]} \pi^\vee, \quad P^+ \curvearrowright \pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F}).$$

$D(\pi)$ is an étale (φ, Γ)-mod over $\mathbb{F}((T))$

$\rightsquigarrow \quad V(\pi) := V(D(\pi))(1)$ Gal rep'n assoc to π.

$\uparrow$ twist by ω

Thm For any $\bar{\rho} : G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$, $\exists !$ sm adm fin-length rep'n $\pi(\bar{\rho})$ of $GL_2(\mathbb{Q}_p)$ over $\mathbb{F}$

s.t. $\cdot$ $V(\pi(\bar{\rho})) \simeq \bar{\rho}$

$\cdot$ $\pi(\bar{\rho})$ has central char $\det(\bar{\rho}) \cdot \omega$

$\cdot$ $\pi(\bar{\rho})$ has no fin-dim'l $GL_2(\mathbb{Q}_p)$-subrep.

(normalization: $\mathrm{rec}(p) = $ geom Frob.)

Rmk $\cdot$ $\pi(\bar{\rho})$ is supersingular $\Leftrightarrow$ $\bar{\rho}$ is irred.

$\cdot$ $\bar{\rho}^{ss} \simeq \chi_1 \oplus \chi_2 \rightsquigarrow \pi(\bar{\rho})^{ss} \simeq \mathrm{Ind}_B^G(\chi_2 \otimes \chi_1\omega^{-1})^{ss} \oplus \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2\omega^{-1})^{ss}$.

Generic condition $\chi_1/\chi_2 \neq \omega^{\pm 1}$ (mod p LLC)

Def'n π = unitary adm residually finite length E-Banach space rep'n of $GL_2(\mathbb{Q}_p)$ with a central char,

$$\pi^\circ = \{ v \in \pi \mid |v| \leq 1 \} \rightsquigarrow V(\pi^\circ) := \varprojlim_n V(\pi^\circ / \varpi^n \pi^\circ)$$

$$V(\pi) := V(\pi^\circ)[\tfrac{1}{p}]$$

Thm For any $\rho : G_{\mathbb{Q}_p} \longrightarrow GL_2(E)$, $\exists !$ unitary adm residually fin-length E-Banach space rep'n $\pi(\rho)$ of $GL_2(\mathbb{Q}_p)$

s.t. $\cdot\ V(\pi(\rho)) \simeq \rho$

$\cdot\ \pi(\rho)^\circ/\varpi \simeq \kappa(\bar\rho)$,

$\cdot\ \pi(\rho)$ has central char $\det\rho\cdot\chi$.

<u>Rmk</u> $\pi(\rho)$ is ss $\iff$ ρ irred.

<u>§2 Deformation theory & Paškūnas modules</u>

<u>Galois side</u> $\bar\rho: G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$ s.t. $\mathrm{End}_{G_{\mathbb{Q}_p}}(\bar\rho) = \mathbb{F}$.

$\overset{\text{Mazur}}{\rightsquigarrow}$ universal deformation $(R_{\bar\rho}, V_{univ})$,

$$R_{\bar\rho} \simeq \mathcal{O}[\![x_1, \cdots, x_5]\!]$$

$$\begin{tikzcd} \zeta: G_{\mathbb{Q}_p} \arrow[rr] \arrow[dr] & & \mathbb{F}^\times \\ & G_{\mathbb{Q}_p}^{ab} \arrow[ur] & \end{tikzcd} \rightsquigarrow R_{\bar\rho}^{\zeta} \text{ parametrizing deformations of } \bar\rho \text{ with det } \zeta$$

$$R_{\bar\rho}^{\zeta} \simeq \mathcal{O}[\![x_1, x_2, x_3]\!].$$

$(R_{\bar\rho}^{\square}, V_{univ}^{\square})$: universal framed deformations

$$R_{\bar\rho}^{\square} \simeq R_{\bar\rho}^{\zeta} \underset{\substack{\uparrow \\ \text{non-canonical.}}}{\hat\otimes_{\mathcal{O}}} \mathcal{O}[\![u, v, z_1, z_2, z_3]\!]$$

<u>$GL_2(\mathbb{Q}_p)$-side</u>

<u>Kisin</u>: Colmez functor extends to the level of deformations

$\mathfrak{C}$: Cat of profinite $\mathcal{O}$-mods M equipped w/ cont right $GL_2(\mathbb{Q}_p)$-actions

s.t. $\cdot$ $GL_2(\mathbb{Z}_p)$-action extends to $\mathcal{O}[\![GL_2(\mathbb{Z}_p)]\!]$-action

$\cdot$ for any $v \in M^\vee = \mathrm{Hom}(M, E/\mathcal{O})$, $\mathcal{O}[GL_2(\mathbb{Q}_p)]$-submod generated by v is of finite length.

$\mathfrak{C}_\zeta :=$ subcat of $\mathfrak{C}$ consisting of objs w/ central char ζ.

$\tilde P_\zeta :=$ universal deformation of $\kappa(\bar\rho)^\vee$ in $\mathfrak{C}_\zeta$.

Kisin's observation $\Rightarrow$ $R_{\bar\rho}^{\zeta}$ naturally acts on $\tilde P_\zeta$.

<u>Thm</u> (Colmez, Paškūnas)

(1) $\widetilde{P}_\xi$ flat over $R^\xi_{\bar\rho}$ and $\widetilde{P}_\xi \otimes_{R^\xi_{\bar\rho}} \mathbb{F} \simeq \kappa(\bar\rho)^\vee$

(2) $\mathrm{End}_{GL_2(\mathbb{Q}_p)}(\widetilde{P}_\xi) \simeq R^\xi_{\bar\rho}$, $\quad V(\widetilde{P}_\xi) \simeq V_{univ}$ as $R^\xi_{\bar\rho}[G_{\mathbb{Q}_p}]$-rep'ns.

For any $x : \mathrm{Spec}\, R^\xi_{\bar\rho}[\tfrac{1}{p}] \longrightarrow \bar{\mathbb{Q}}_p$,

$$V(\widetilde{P}_\xi \otimes_{R^\xi_{\bar\rho},x} \bar{\mathbb{Q}}_p) \simeq V_{univ,x}.$$

(3) $\widetilde{P}_\xi$ is the proj envelope of $\kappa(\bar\rho)^\vee$ in $\mathcal{C}_\xi$,

$$\bar\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \text{ non-split.} \qquad \kappa(\bar\rho) = (\pi_1 - \pi_2).$$

In this case, $\widetilde{P}_\xi$ is also the proj envelope of $\pi_1^\vee$.

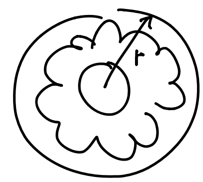
<u>§3 Trianguline deformation space</u>

Robba ring $\mathcal{R} = \{ f(T) = \sum_{i\in\mathbb{Z}} a_i T^i \mid a_i \in E,\ f$ convergent on $1 > |T| \geq r \}$.

$\rightsquigarrow$ $\varphi, \Gamma \curvearrowright \mathcal{R}$ $\quad$ (φ,Γ)-mod over $\mathcal{R}$.

$\mathcal{E}^\dagger = \{ f \in \mathcal{R} \mid \{|a_i|\}$ is bounded$\}$.

$\mathcal{R} \supset \underset{\varphi,\Gamma}{\circlearrowright} \subset \mathcal{E}$



<u>Thm</u> (Cherbonnier-Colmez, Kedlaya)

$\exists$ rank-preserving equiv of cats

$\{$étale (φ,Γ)-mods$/\mathcal{R}\} \longleftrightarrow \{G_{\mathbb{Q}_p}$-rep'ns over $E\}$.

$$D^\dagger_{rig}(V) \longleftarrow\!\!\shortmid V$$

<u>Def'n</u> A rank d (φ,Γ)-mod D over $\mathcal{R}$ is called <u>trianguline</u> if $\exists$ a filtration

$$0 = \mathrm{Fil}^0 D \subset \mathrm{Fil}^1 D \subset \cdots \subset \mathrm{Fil}^d D = D$$

of (φ,Γ)-submods s.t. $\mathrm{Fil}^{i+1}/\mathrm{Fil}^i$ is a rank 1 (φ,Γ)-mod $\mathcal{R}(\delta_i)$.

A trianguline (φ,Γ)-mod $\rightsquigarrow (\delta_1, \delta_2, \dots, \delta_d)$.

Thm (Kisin) f finite slope overconvergent p-adic modular forms.

Then $D^{+}_{rig}(\rho_f)$ is trianguline (ρ_f is a trianguline rep'n).

Thm V = 2-dim'l rep'n of $G_{\mathbb{Q}_p}$. Then

V trianguline $\Longleftrightarrow$ V crystabelian

i.e. V becomes crystalline after an abelian ext'n of $\mathbb{Q}_p$.

Def D rank 2 trianguline (φ, Γ)-mods $/\mathcal{R}$.

$\rightsquigarrow$ $0 \to \mathcal{R}(\delta_1) \to D \to \mathcal{R}(\delta_2) \to 0$.

Say D is étale if $\cdot$ $v_p(\delta_1(p)) + v_p(\delta_2(p)) = 0$,

$\nexists$ rank 1 (φ, Γ)-submod of D has negative slope.

In particular, $v_p(\delta_1(p)) \geq 0$.

§4 Trianguline deformation space à la BHS

T = rigid analytic space parametrizing cont chars of $(\mathbb{Q}_p^\times)^2 \to E^\times$

$= (\mathbb{G}_m^{rig})^2 \times (\mathrm{Spf}\, \mathcal{O}[\![(\mathbb{Z}_p^\times)^2]\!])^{rig}$.

$T_{reg} = \{(\delta_1, \delta_2) \in T \mid (\delta_1/\delta_2)^{\pm 1} \neq x^n \cdot \chi,\ n \geq 0\}$ $\quad$ ($\dim H^1(\delta_1/\delta_2) = 2$.)

generic condition.

$\mathcal{X}^{\square}_{\bar{\rho}} = (\mathrm{Spf}\, R^{\square}_{\bar{\rho}})^{rig}$, $\mathcal{X}^{\square}_{\bar{\rho}}$ is of dim 8 $/E$.

Def $U^{\square, tri}_{\bar{\rho}, reg}$ = set of pts $(x, \delta_1, \delta_2) \in \mathcal{X}^{\square}_{\bar{\rho}} \times T_{reg}$

s.t. $0 \to \mathcal{R}(\delta_1) \to D_{rig}(V_x) \to \mathcal{R}(\delta_2) \to 0$.

Trianguline deformation space

$\mathcal{X}^{\square, tri}_{\bar{\rho}} :=$ Zariski closure of $U^{\square, tri}_{\bar{\rho}, reg}$ in $\mathcal{X}^{\square}_{\bar{\rho}} \times T$.

<u>Thm</u> · $U^{\square,\mathrm{tri}}_{\bar{p},\mathrm{reg}}$ is the set of closed pts of a Zariski open and dense subspace $\mathcal{U}^{\square,\mathrm{tri}}_{\bar{p},\mathrm{reg}}$ of $X^{\square,\mathrm{tri}}_{\bar{p}}$.

· $X^{\square,\mathrm{tri}}_{\bar{p}}$ is equidim'l of dim 7.