Lecture 9: Bootstrapping argument

"Automorphic side"		"Galois side"
local ghost thm	Paškūnas module; Emerton's Jacquet functor $\rightsquigarrow$	slopes on trianguline deformation space
"global" ghost thm of Bergdall-Pollack	$\leftsquigarrow$ "bootstrapping"	

<u>Setup</u> $p \geq 11$, $2 \leq a \leq p-5$. (will assume $b=0$ for simplicity.)

$F/\mathbb{Q}_p \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) = \mathbb{F}$.

$$\omega_1 : \mathrm{Gal}_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p) \simeq \mathbb{F}_p^\times$$

$$\mathrm{unr}(\bar\alpha) : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{Gal}_{\mathbb{F}_p} \to \mathbb{F}^\times$$
$$\mathrm{geomFr}_p \mapsto \bar\alpha \in \mathbb{F}^\times.$$

$$\bar r_p = \begin{pmatrix} \mathrm{unr}(\bar\alpha_1)\cdot\omega_1^{a+1} & * \\ 0 & \mathrm{unr}(\bar\alpha_2) \end{pmatrix} : \mathrm{Gal}_{\mathbb{Q}_p} \to GL_2(\mathbb{F}).$$

Always denote $\bar\rho : I_{\mathbb{Q}_p} \to GL_2(\mathbb{F})$

$$\bar\rho = \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & 1 \end{pmatrix}$$ comes from an ext'n of rep'ns of $\mathrm{Gal}_{\mathbb{Q}_p}$.

$\bar\rho^{ss} := \omega_1^{a+1} \oplus 1$.

<u>Have explained</u>: $\mathcal{X}_{\bar r_p}^{\square,\mathrm{tri}}$:= trianguline deform space

$(x, \delta_1, \delta_2) \in \mathcal{X}_{\bar r_p}^{\square,\mathrm{tri}}$ s.t. $0 \to \mathcal{R}(\delta_1) \to D_{\mathrm{rig}}(V_x) \to \mathcal{R}(\delta_2) \to 0$.

$$\mathcal{X}_{\bar r_p}^{\square,\mathrm{tri}} \longrightarrow \mathcal{W}^{(\varepsilon)} \qquad \Delta = \mathbb{F}_p^\times \xhookrightarrow{\omega} \mathbb{Z}_p^\times.$$

$(x, \delta_1, \delta_2) \longmapsto \varepsilon = \delta_2|_\Delta \times \delta_1|_\Delta \cdot \omega^{-1}$ (relevant to $\bar\rho$).

(<u>Normalizations</u> $w_\star = (\delta_1\delta_2^{-1}\chi_{\mathrm{cycl}}^{-1})(\exp(p)) - 1$.

$w_k = \exp(pk) - 1$.)

<u>Theorem</u> Suppose $(x, \delta_1, \delta_2) \in \mathcal{X}^{\square, \mathrm{tri}}_{\bar{r}_p}$.

(1) If $v_p(\delta_1(p)) = -v_p(\delta_2(p)) > 0$

$\Rightarrow v_p(\delta_1(p))$ is a slope of

$$NP(G_{\bar{p}}^{(\varepsilon)}(w_\star, -)), \quad w_\star = (\delta_1 \delta_2^{-1} \chi_{cycl}^{-1})(\exp(p)) - 1.$$

(2) If $v_p(\delta_1(p)) = 0$ then $\varepsilon = \begin{cases} 1 \times \omega^a \\ \omega^{a+1} \times \omega^{-1} \text{ \& } \bar{r}_p \text{ is split.} \end{cases}$

(3) If $v_p(\delta_1(p)) = \frac{k}{2} - 1$ and $w_\star = w_k$ for an integer k,

then $\delta_1(p) = p^{k-2}\delta_2(p)$.

} standard (φ, Γ)-mod theory.

* Conversely, given any slope in $NP(G_{\bar{p}}^{(\varepsilon)}(w_\star, -))$, $\exists (x, \delta_1, \delta_2)$ as above

<u>Proof</u> Only in the case when $\bar{r}_p$ is nonsplit

$$\bar{r}_p = \underbrace{\mathrm{unr}(\bar{\alpha}_1) \cdot \omega_1^{a+1}}_{} \text{ --- } \underbrace{\mathrm{unr}(\bar{\alpha}_2)}_{}$$

char χ_1 of $\mathbb{Q}_p^\times$ $\qquad \chi_2 : \mathbb{Q}_p^\times \xrightarrow{v_p(-)} \mathbb{Z} \longrightarrow \mathbb{F}^\times$

$\qquad\qquad 1 \longmapsto \bar{\alpha}_2$.

$$\bar{\pi}_1 = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\bar{\chi}_2 \otimes \bar{\chi}_1 \cdot \bar{\chi}_{cycl}^{-1}), \quad \bar{\pi}_2 = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\bar{\chi}_1 \otimes \bar{\chi}_2 \cdot \bar{\chi}_{cycl}^{-1})$$

$\bar{\pi}(\mathbb{Q}_p) = \bar{\pi}_1 - \bar{\pi}_2$.

Fix a central char $\zeta : \mathbb{Q}_p^\times \to \mathcal{O}^\times$ s.t. $\zeta|_\Delta \equiv \omega^a \bmod \varpi$.

<u>Main subject</u> In $\mathcal{C}_\zeta$, $\tilde{P}_\zeta \twoheadrightarrow \pi_1^\vee$

$R_{\bar{r}_p}^\zeta \circlearrowright$ " proj envelope of $\pi_1^\vee$.

Put $\tilde{P}^\square := \tilde{P}_\zeta \boxtimes \mathbb{1}_{tw}$

$R_{\bar{r}_p}^\square \circlearrowright \quad GL_2(\mathbb{Q}_p) \circlearrowright \quad$ central twist

a char $\mathbb{Q}_p^\times \longrightarrow \mathcal{O}[\![u, v]\!]$

$p \longmapsto 1+u$

$\exp(p) \longmapsto 1+v$

<u>Key</u> (Hu–Paškūnas) $\exists x \in \mathfrak{m}_{R_{\bar{r}_p}^\square} \setminus \mathfrak{m}^2$, s.t. as an $\underbrace{\mathcal{O}[\![u, x, z_1, z_2, z_3]\!]}_{=: S^\square}$, $[\![GL_2(\mathbb{Z}_p)]\!]$-mod,

$\tilde{P}^\square$ is the proj envelope of $\mathrm{Sym}^a(\mathbb{F}^{\oplus 2})$.

$$\begin{tikzcd} R^{\square}_{\bar{r}_p} \curvearrowright \tilde{P}^{\square} \arrow[d, dash] \\ S^{\square} \end{tikzcd}$$

Rmk for any evaluation $s^*: S \to \mathcal{O}'$, $w \mapsto w_0$, $x \mapsto x_0$, $\mathcal{O}'/\mathcal{O}$ fin ext'n,

$s^*\tilde{P}^{\square} := \boxed{\tilde{P}^{\square} \otimes_{S, s^*} \mathcal{O}'}$ is a primitive K_p-proj augmented module of type $\bar{\rho}$.

can apply local ghost thm to this

- Put $\Pi^{\square} := \mathrm{Hom}^{\mathrm{cont}}(\tilde{P}^{\square}, E)$.

Define $\mathcal{M}^{\square} := \mathrm{swap}^*\left((J_B(\Pi^{\square})^{S^{\square}\text{-an}})'_b \right)$

$\mathcal{M}^{\square}$ — $\mathcal{T}$; J_B: Emerton's Jacquet functor

$\mathrm{Hom}((\mathbb{Q}_p^\times)^2, \mathbb{C}_p^\times)$ " $\mathrm{swap}: \mathcal{T} \longrightarrow \mathcal{T}$

$(\delta_1, \delta_2) \longmapsto (\delta_2, \delta_1)$

$\mathcal{E}ig(\rho^{\square}) = \mathrm{Supp}(\mathcal{M}^{\square})$ over $R^{\square}_{\bar{r}_p} \times \mathcal{T} \times \mathcal{Y}^{\square}$, $\mathcal{Y}^{\square} = (\mathrm{Spf}\, S^{\square})^{\mathrm{rig}}$.

Key input (Breuil-Ding, Breuil-Hellman-Schraen)

$\mathcal{E}ig(\rho^{\square})^{(\mathrm{red})} \cong X^{\square,\mathrm{tri}}_{\bar{r}_p}$ ("global trianguline + density of classical pts")

slopes here are governed by ghost series

$$\begin{tikzcd} & & \mathcal{E}ig(\rho^{\square})^{(\mathrm{red})} \arrow[dl] & X^{\square,\mathrm{tri}}_{\bar{r}_p} \arrow[d] \\ \boxed{\mathrm{Spc}(s^*\tilde{P}^{\square})} \arrow[r, hook] \arrow[d] \arrow[drr, phantom, "\square"] & \mathrm{Spc}(\tilde{P}^{\square}) \arrow[rr] & & \mathcal{Y}^{\square} \times \mathcal{T} \arrow[r, "\delta_2(p)"] \arrow[d] & \mathbb{G}_m^{\mathrm{rig}} \\ \mathcal{W} \arrow[rrr, "s"] & & & \mathcal{Y}^{\square} \times \mathcal{W} \end{tikzcd}$$

§ Bootstrapping

* Local-global Compatibility @ p $\qquad K_p = GL_2(\mathbb{Z}_p)$.

Let $\tilde{H}$ be a K_p-proj augmented mod

(i.e. fin proj right $\mathcal{O}[\![K_p]\!]$-mod whose K_p-action extends to a $GL_2(\mathbb{Q}_p)$-action.)

s.t. $\forall \bar\alpha_1 \in \Delta$, $\begin{pmatrix}\bar\alpha_1 & \\ & \bar\alpha\end{pmatrix}$ acts on $\tilde H$ by $\bar\alpha^a$.

Fix $\varepsilon: \Delta^2 \to \mathbb{F}^\times$ relevant, i.e. $\varepsilon(\bar\alpha, \bar\alpha) = \bar\alpha^a$.

$\varepsilon = \omega^{-s} \times \omega^{a+s}$ for $s \in \{0, \dots, p-2\}$.

For $k = k_\varepsilon := a + 2s + 2 \mod p-1$ (and $k \geq 2$).

$$T_p, S_p \curvearrowright S_k^{\mathrm{univ}}(\omega^{-s}) := \mathrm{Hom}_{\mathcal{O}[[K_p]]}(\tilde H, \mathcal{O}[z]^{\deg \leq k-2} \otimes \omega^{-s} \circ \det)$$

$$S_p(\varphi)(x) := \varphi\left(x \begin{pmatrix} p^{-1} & \\ & p^{-1}\end{pmatrix}\right).$$

The eigenvalues of T_p, S_p are expected to correspond to sth on Gal side:

$R_{\bar r_p}^{\square, 1-k, \omega^{-s}}$:= crystalline framed deform space of $\bar r_p$ of HT wts $(1-k, 0)$ and $\mathrm{Gal}_{\mathbb{Q}_p}$ acts on $\mathbb{D}_{pcrys}(-)$ by ω^{-s}.

$$\begin{array}{ll} \rightsquigarrow & \mathbb{D}_{pcrys}(\mathcal{V}_{1-k}) \circlearrowleft \text{ crystalline Frob } \phi \\ \text{loc free of rk 2} & \big| \\ & \mathcal{X}_{\bar r_p}^{\square, 1-k, \omega^{-s}} \end{array}$$

(six authors) $\Rightarrow$ $\exists$ elements $s_p, t_p \in R_{\bar r_p}^{\square, 1-k, \omega^{-s}}[\frac{1}{p}]$.

s.t. $\det(\phi) = p^{k-1} \cdot s_p^{-1}$, $\mathrm{tr}(\phi) = s_p^{-1} t_p$.

<u>Def'n</u> An $\mathcal{O}[[K_p]]$-proj arith modular of type $\bar r_p$ is an $\mathcal{O}[[K_p]]$-proj augmented mod $\tilde H$ equipped with a cont. left action of $R_{\bar r_p}^{\square}$.

s.t. ① left $R_{\bar r_p}^{\square}$-action and right $GL_2(\mathbb{Q}_p)$-action commute

② $\tilde H$ as a right $\mathcal{O}[[K_p]]$-mod is isom to

- $\mathrm{Proj}_{\mathcal{O}[[K_p]]}(\mathrm{Sym}^a)^{\oplus m(\tilde H)}$ if $\bar r_p$ is non-split (Serre wt = $\mathrm{Sym}^a \mathbb{F}^2$).
- $\mathrm{Proj}(\mathrm{Sym}^a)^{\oplus m'(\tilde H)} \oplus \mathrm{Proj}(\mathrm{Sym}^{p-3-a} \otimes \det^{a+1})^{\oplus m''(\tilde H)}$ if $\bar r_p$ is split (Serre wts Sym^a, $\mathrm{Sym}^{p-3-a} \otimes \det^{a+1}$).

· $m(\tilde H) = m'(\tilde H) + m''(\tilde H)$

③ $\forall \varepsilon = \omega^{-s} \times \omega^{a+s}$ relevant, $k \equiv a+2s+2 \mod p-1$, $k \geq 2$

$R^{\square}_{\bar{r}_p}$ - action on $S^{\mathrm{Iw}}_k(\omega^{-s})$ factors through $R^{\square, 1-k, \omega}_{\bar{r}_p}$.

$$\begin{tikzcd} t_p, s_p \in R^{\square, 1-k, \omega^{-s}}_{\bar{r}_p} \arrow[loop below] & R^{\square}_{\bar{r}_p} \arrow[l, two heads] \arrow[loop below] \end{tikzcd}$$

$$S^{\mathrm{Iw}}_k(\omega^{-s}) := \mathrm{Hom}_{\mathcal{O}[\![K_p]\!]}(\widetilde{H}, \mathcal{O}[z]^{\deg \leq k-2} \otimes \omega^{-s} \circ \det),$$

$\circlearrowleft S_p, T_p$

<u>Example</u> "Essentially" for any $G/\mathbb{Q}$ s.t. $G_{\mathbb{Q}_p} \approx GL_2(\mathbb{Q}_p) \times H$

$\left(\begin{array}{l} \text{e.g. } G = \mathrm{Res}_{F/\mathbb{Q}} GL_2, \\ \quad F \text{ totally real} \end{array}\right.$ $\quad \begin{array}{c} F \\ | \\ \mathbb{Q} \end{array} \quad \begin{array}{c} \mathfrak{p}_1 \cdots \cdots \\ \backslash // \\ p \end{array} \quad F_{\mathfrak{p}_1} = \mathbb{Q}_p. \Big)$

$R_{\bar{r}_p}$ $\circlearrowright$

$$\widetilde{H} := \varprojlim_N H_{\mathrm{mid}}(\mathrm{Sh}_G(K^p \cdot K_H \cdot (1 + p^N M_2(\mathbb{Z}_p)))(\mathbb{C})^{\mathrm{an}}, \mathbb{Z}_p)_{\mathfrak{m}_{\bar{r}}}$$

$*$ for some $\bar{r}$ irred + "large image" (to apply Caraiani-Scholze.)

<u>Or</u>, a patched version of this!

$$\begin{tikzcd} R^{\square}_{\bar{r}_p} \curvearrowright \widetilde{H}_\infty \arrow[d, dash] & \widetilde{H}_\infty \otimes_{J_\infty, \underline{z}^*} \mathcal{O}' \curvearrowleft R^{\square}_{\bar{r}_p}. \arrow[d, dash] \\ J_\infty = \mathcal{O}[\![z_1, \cdots, z_g]\!] \arrow[r, "\underline{z}^*"] & \mathcal{O}' \end{tikzcd}$$

<u>Rmk 1</u> What if $\bar{r}$ is not irred?

(Some combinatorics to be done,
see Diao-Yao's recent work)

<u>Rmk 2</u> Why Paškūnas mod but not patched module of six author?

<u>Thm</u> Let $\widetilde{H}$ be an $\mathcal{O}[\![K_p]\!]$-proj arith mod of type $\bar{r}_p$ and multiplicity $m(\widetilde{H})$.

Let $C^{(\varepsilon)}_{\widetilde{H}}(w,t) :=$ char power series of $U_p \curvearrowright S^{p\text{-adic}, (\varepsilon)}_{\widetilde{H}}$.

Then for any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, $NP(C^{(\varepsilon)}_{\widetilde{H}}(w_\star, -)) = NP(G^{(\varepsilon)}_{\bar{\rho}}(w_\star, -))$

stretched in both x-, y-directions $m(\widetilde{H})$ times.

(except for the ordinary part.)

<u>Proof</u>

$$\text{Length of ord part} = \begin{cases} m(\widetilde{H}) & \text{if } \bar{r}_p \text{ non-split and } \varepsilon = 1 \times \omega^a \\ m'(\widetilde{H}) & \text{if } \bar{r}_p \text{ split and } \varepsilon = 1 \times \omega^a \\ m''(\widetilde{H}) & \text{if } \bar{r}_p \text{ split and } \varepsilon = \omega^{a+1} \times \omega^{-1}. \\ 0 & \text{o/w.} \end{cases}$$

density of classical pts

$$\begin{tikzcd}
\mathcal{X}^{\square,\mathrm{tri}}_{\bar{r}_p} \arrow[d, hook] \arrow[r, phantom, "\supseteq"] & \mathcal{E}ig^{(\varepsilon)}(\widetilde{H}) \arrow[r, "\text{``finite''}"] \arrow[d] & \mathrm{Spc}^{(\varepsilon)}(\widetilde{H}) \arrow[d] \arrow[drr, dashed, "\mathrm{wt}"] \arrow[loop above, "R^{\square}_{\bar{r}_p}"] & & \\
\mathcal{X}^{\square}_{\bar{r}_p} \times \mathcal{T} \arrow[r, leftrightarrow] & \mathcal{X}^{\square}_{\bar{r}_p} \times \mathcal{W}^{(\varepsilon)} \times \mathbb{G}_m^{\mathrm{rig}} \arrow[r] \arrow[l, bend left] & \mathcal{W}^{(\varepsilon)} \times \mathbb{G}_m^{\mathrm{rig}} \arrow[rr] & & \mathcal{W}^{(\varepsilon)}
\end{tikzcd}$$

$\mathrm{Spc}^{(\varepsilon)}(\widetilde{H}) =$ zero locus of $C^{(\varepsilon)}(w,t)$.

at least the reduced part

need to choose this normalization here.

$\Rightarrow$ Pointwise, slopes on $\mathrm{Spc}^{(\varepsilon)}(\widetilde{H})$ are the slopes of ghost series

but not sure about multiplicity yet.

Fix ε relevant. $\forall n \in \mathbb{N}$, define

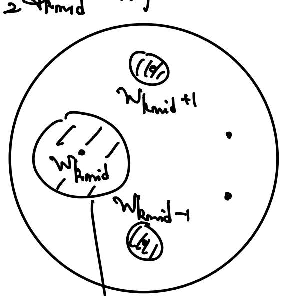
$$\mathrm{Vtx}_n := \{ w_\star \in \mathfrak{m}_{\mathbb{C}_p} : (n, v_p(g_n(w_\star))) \text{ is a vertex of } NP(G_{\bar{p}}, (w_\star, -)) \}$$

$$= \mathcal{W}^{(\varepsilon)} \setminus \bigsqcup_k \{ w_\star \in \mathfrak{m}_{\mathbb{C}_p} \mid v_p(w_\star - w_k) \geq \Delta_{k, |\frac{1}{2} d_k^{\mathrm{Iw}} - n| + 1} - \Delta_{k, |\frac{1}{2} d_k^{\mathrm{Iw}} - n|} \}$$

quasi-Stein, irred.

$$= \bigcup_{\delta \to 0^+} \overline{\mathrm{Vtx}_n^{\delta}} \quad \overline{\mathrm{Vtx}_n^{\delta}} = \left\{ w_\star \in \mathfrak{m}_{\mathbb{C}_p} \;\middle|\; \begin{array}{l} v_p(w_\star) \geq \delta, \\ v_p(w_\star - w_k) \leq \cdots - \delta, \ \forall k \end{array} \right\}$$

$\left(\frac{1}{2} d_{k_{\mathrm{mid}}}^{\mathrm{Iw}} = n \right)$



$v_p(-) = \frac{a+2}{2}, \frac{p-4-a}{2}$

<u>Upshot</u> at each pt $w_\star \in \mathrm{Vtx}_n^{\delta}$,

the left slope at $x = n$ of $NP(G_{\bar{p}})$

$\leq$ (the right slope at $x = n$ of $NP(G_{\bar{p}})$) $- \epsilon(\delta)$.

Upshot

$$\mathrm{Spc}(\tilde{H})_n^{\delta} := \left\{ (w_*, a_p) \in \mathrm{Spc}(\tilde{H}) \;\middle|\; \begin{array}{l} w_* \in \mathcal{U}_{tx_n}^{\delta} \\ -v_p(a_p) \le \text{left slope at } x=n \text{ of } NP(G) \end{array} \right\}$$

strict relative to $\mathcal{U}_{tx_n}$ $\;\downarrow\; \|$

$$\mathrm{Spc}(\tilde{H})_n^{\delta,+} := \left\{ (w_*, a_p) \in \mathrm{Spc}(\tilde{H}) \;\middle|\; \begin{array}{l} w_* \in \mathcal{U}_{tx_n}^{\delta} \\ -v_p(a_p) \le \text{right slope at } x=n \text{ of } NP(G) + \epsilon(\delta) \end{array} \right\}$$

Kiehl's argument $\Longrightarrow$ $\mathrm{wt}_*(\mathrm{Spc}(\tilde{H})_n^{\delta})$ = finite over $\mathcal{U}_{tx_n}^{\delta}$, flat by construction $\Big\}$ $\Rightarrow$ Constant degree.

Technical lemma (next lecture)

$\forall k$, $n = d_k^{\mathrm{Iw}}(\varepsilon \cdot (1 \times \omega^{2-k}))$ (usually essential Iwp-level)

$\Rightarrow$ $(n, v_p(g_n(w_k)))$ is a vertex for $NP(G)$.

By dim formula, done!