Elementary Number Theory

PROBLEM SET 1

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Problem 1. Let a, b be positive integers with $\gcd(a, b) = 1$.

(a) Prove that $ab - a - b$ cannot be written as $ax + by$ where x, y are non-negative integers.
(b) Prove that for any integer $n > ab - a - b$, there exist non-negative integers x, y such that $n = ax + by$.

Problem 2. Let $\mathbb{Z}_{\geqslant 0}$ denote the set of non-negative integers. Let $f : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ be a function such that

$$f(b, a) = f(a, b) = f(b - a, a)$$

for any integers $0 \leqslant a \leqslant b$. Prove that

$$f(a, b) = f(\gcd(a, b), 0)$$

for any $a, b, \in \mathbb{Z}_{\geqslant 0}$.

Problem 3. Let a, m, n be positive integers. Prove that

$$\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1.$$

Problem 4. Let x, y, z be positive integers such that $\gcd(x, y, z) = 1$ and

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Prove that $x + y$ is a perfect square.

Problem 5. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers such that

$$\gcd(a_m, a_n) = \gcd(m, n)$$

for any $m, n \in \mathbb{N}$ that are distinct. Prove that $a_m = m$ for any $m \in \mathbb{N}$.

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