Problem Set 2 Solutions

1. (a) Write $n = p_1^{k_1} \cdots p_r^{k_r}$ in prime decomposition.

$\forall i = 1, \cdots, r$, define

$$s_i = \begin{cases} k_i, & \text{if } 2 \mid p_i \\ k_i - 1, & \text{if } 2 \nmid p_i \end{cases}$$

i.e. $s_i = 2 \cdot \lfloor \frac{k_i}{2} \rfloor$. Take $t_0 = p_1^{s_1} \cdots p_r^{s_r} \Rightarrow t_0$ perfect square.

Write $t_0 = t^2$, $t = p_1^{\lfloor k_1/2 \rfloor} \cdots p_r^{\lfloor k_r/2 \rfloor}$, $s = \frac{n}{t^2}$. Then $n = st^2$

with $v_{p_i}(s) = 0$ or 1 ($\Rightarrow s$ square free.).

(b) Note that

$$\#\{\text{squares} \leq x\} = \lfloor \sqrt{x} \rfloor \leq \sqrt{x},$$

Also, let $p_1, \cdots, p_{\pi(x)}$ be all primes $\leq x$.

Then any square-free integer generated by them must

equal to $p_1^{r_1} \cdots p_{\pi(x)}^{r_{\pi(x)}}$ for each $r_i \in \{0, 1\}$

$\Rightarrow$ they generate $2^{\pi(x)}$ square-free integers.

$\Rightarrow$ $\pi(x) = \log_2(\#\text{ square-free integers generated by primes} \leq x)$

$\geq A \cdot \log_2(x)$ for some const $A > 0$.

So $\pi(x) \geq C \cdot \log x$ by taking $C = \frac{A}{\log 2} > 0$.

2. $\forall\, 0 < j \le k$, we have

$$\left\lfloor \frac{n}{p^j} \right\rfloor = a_k p^{k-j} + a_{k-1} p^{k-1-j} + \cdots + a_j + \left\lfloor \frac{a_{j-1}}{p} + \frac{a_{j-2}}{p^2} + \cdots + \frac{a_0}{p^j} \right\rfloor$$

where $0 \le \frac{a_{j-1}}{p} + \cdots + \frac{a_0}{p^j} = \frac{1}{p^j}\left(a_{j-1} p^{j-1} + \cdots + a_0\right)$

$$\le \frac{p-1}{p^j} \cdot (1 + p + \cdots + p^{j-1}) \quad \text{as each } a_i \le p-1$$

$$= \frac{p-1}{p^j} \cdot \frac{p^j - 1}{p-1} = \frac{p^j - 1}{p^j} < 1$$

$$\Rightarrow \left\lfloor \frac{n}{p^j} \right\rfloor = a_k p^{k-j} + a_{k-1} p^{k-1-j} + \cdots + a_j .$$

So $v_p(n!) = \sum_{j=1}^{k} \left\lfloor \frac{n}{p^j} \right\rfloor$ by definition of $v_p(n!)$

$$= \sum_{j=1}^{k} \left(a_k p^{k-j} + a_{k-1} p^{k-1-j} + \cdots + a_j\right)$$

$$= a_k(p^{k-1} + \cdots + 1) + a_{k-1}(p^{k-2} + \cdots + 1) + \cdots + a_1$$

$$= a_k \frac{p^k - 1}{p-1} + a_{k-1} \frac{p^{k-1} - 1}{p-1} + \cdots + a_1 \cdot \frac{p-1}{p-1}$$

$$= \frac{1}{p-1}\Big((a_k p^k + a_{k-1} p^{k-1} + \cdots + a_0) - (a_k + a_{k-1} + \cdots + a_0) \Big)$$

$$= \frac{1}{p-1}\left(n - S_p(n)\right).$$

3. By Problem 2, $a!b! \mid n!$

$\Rightarrow v_2(a!) + v_2(b!) \leq v_2(n!)$

$\Rightarrow a - S_2(a) + b - S_2(b) \leq n - S_2(n)$

$\Rightarrow a + b - n \leq S_2(a) + S_2(b) - S_2(n)$.

So it suffices to show RHS $\leq 1 + 2\frac{\log n}{\log 2} = 1 + 2\log_2 n$.

Write $n = n_0 + 2n_1 + \cdots + 2^k n_k$, with each $n_i \in \{0,1\}$, $n_k \neq 0$.

$\Rightarrow \log_2 n \in [k, k+1) \Rightarrow 1 + 2\log_2 n \geq 1 + 2k$.

Also, $S_2(n) = n_0 + n_1 + \cdots + n_k \in [1, k+1]$ as $n_k \neq 0$.

And $a!b! \mid n! \Rightarrow a, b < n$

$\Rightarrow S_2(a), S_2(b) \leq k+1$ as each coefficient is 0 or 1.

$\Rightarrow S_2(a) + S_2(b) - S_2(n) \leq k+1 + k+1 - 1 = 2k+1$

which is as desired.

4. Induct on n.

(1) $n=1$, the assertion is trivial.

(2) Suppose $\forall m \in \{1, \dots, n\}$, $m\binom{n}{m} \mid \operatorname{lcm}(1, \dots, n)$.

For $n+1$, if $m = n+1$, then

$$m\binom{n+1}{m} = n+1 \mid \operatorname{lcm}(1, \dots, n+1).$$

Assume $m \leq n$ now. Have

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}.$$

By inductive hypothesis,

$$m\binom{n}{m},\ (m-1)\binom{n}{m-1} \mid \operatorname{lcm}(1, \dots, n)$$

$$\Rightarrow \quad m\binom{n}{m} + (m-1)\binom{n}{m-1} \mid \operatorname{lcm}(1, \dots, n+1).$$

Apparently, $\binom{n}{m-1} \mid \operatorname{lcm}(1, \dots, n)$ by hypothesis again.

$$\Rightarrow \quad m\binom{n+1}{m} = m\binom{n}{m} + (m-1)\binom{n}{m-1} + \binom{n}{m-1}$$

divides $\operatorname{lcm}(1, \dots, n+1)$.

So we complete the induction.

5. (a) We have $\gcd(n, n+1) = 1$, $\binom{2n+1}{n} = \binom{2n+1}{n+1}$.

So the condition implies

$$n(n+1)\binom{2n+1}{n} \,\Big|\, N.$$

(b) By Problem 4,

$$n\binom{2n+1}{n} \,\Big|\, \mathrm{lcm}(1, \cdots, 2n+1), \quad (n+1)\binom{2n+1}{n+1} \,\Big|\, \mathrm{lcm}(1, \cdots, 2n+1).$$

So (a) $\Rightarrow$ $n(n+1)\binom{2n+1}{n} \,\Big|\, \mathrm{lcm}(1, \cdots, 2n+1)$.

It suffices to show $n(n+1)\binom{2n+1}{n} \geq n4^n$

$$\Leftrightarrow (n+1)\binom{2n+1}{n} \geq 2^{2n} = (1+1)^{2n}$$

But $(1+1)^{2n} = \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n}$

$$\leq (n+1)\binom{2n}{n} < (n+1)\binom{2n+1}{n}.$$

So we get $n \cdot 4^n \leq \mathrm{lcm}(1, \cdots, 2n+1)$.

(c) (1) $2 \nmid m$, write $m = 2n+1 \geq 7$ ($\Rightarrow n \geq 3$)

(b) $\Rightarrow$ $\mathrm{lcm}(1, \cdots, m) \geq n \cdot 2^{2n} = n \cdot 2^{m-1} \geq 2^m$ as $n \geq 3$.

(2) $2 \mid m$, write $m = 2n \geq 7$ ($\Rightarrow n \geq 4$).

Problem 4 $\Rightarrow$ $n\binom{2n}{n} \,\Big|\, \mathrm{lcm}(1, \cdots, 2n)$.

It suffices to show $4^n < n\binom{2n}{n}$, $\forall n \geq 4$.

Check: (i) $n = 4$: clear.

(ii) Suppose true for $n = k$. For $n = k+1$,

Known: $4^k < k\binom{2k}{k}$

So $\quad 4^{k+1} < (k+1)\binom{2(k+1)}{k+1}$

$$\Leftarrow 4 < (k+1)\binom{2(k+1)}{k+1} \Big/ k\binom{2k}{k}$$

$$= (k+1) \cdot \frac{(2k+2)!}{(k+1)!^2} \cdot \frac{1}{k} \cdot \frac{k!^2}{(2k)!}$$

$$= 2 \cdot \frac{2k+1}{k} = 4 + \frac{2}{k}, \text{ which is true.}$$

So we are done.