代数换元

例1 (IMO 2001, P2) $a,b,c>0$,

证明: $\dfrac{a}{\sqrt{a^2+8bc}}+\dfrac{b}{\sqrt{b^2+8ca}}+\dfrac{c}{\sqrt{c^2+8ab}}\geq 1$.

解答 作换元: $\dfrac{a}{\sqrt{a^2+8bc}}=x$, $\dfrac{b}{\sqrt{b^2+8ca}}=y$, $\dfrac{c}{\sqrt{c^2+8ab}}=z$, $x,y,z\in(0,1)$.

记意: $\dfrac{a^2}{8bc}=\dfrac{x^2}{1-x^2}$, $\dfrac{b^2}{8ca}=\dfrac{y^2}{1-y^2}$, $\dfrac{c^2}{8ab}=\dfrac{z^2}{1-z^2}$.

$$\Rightarrow \quad \frac{1}{512}=\prod_{cyc}\frac{x^2}{1-x^2}.$$

原题 $\Longleftrightarrow$ $0<x,y,z<1$, $(1-x^2)(1-y^2)(1-z^2)=512(xyz)^2$.

证明 $x+y+z\geq 1$.

反证: 设 $x+y+z<1$.

$$\Rightarrow \quad (1-x^2)(1-y^2)(1-z^2)\underset{\text{齐次化}}{>}\prod_{cyc}\left((x+y+z)^2-x^2\right)$$

$$=\prod_{cyc}(y^2+z^2+2xy+2yz+2zx)$$

$$=\prod_{cyc}(x+x+y+z)(y+z).$$

$$\geq\prod_{cyc}4(x^2yz)^{1/4}\cdot 2(yz)^{1/2}$$

$$=512x^2y^2z^2. \quad \text{矛盾.} \qquad \square$$

例2 (IMO 1995, P2) $a,b,c>0$, $abc=1$.

证明: $\displaystyle\sum_{cyc}\frac{1}{a^3(b+c)}\geq\frac{3}{2}$.

解答 $a=\dfrac{1}{x}$, $b=\dfrac{1}{y}$, $c=\dfrac{1}{z}$, $xyz=1$.

原式 $\Longleftrightarrow$ $\displaystyle\sum_{cyc}\frac{x^3yz}{y+z}=\sum_{cyc}\frac{x^2}{y+z}\geq\frac{3}{2}$.

Cauchy-Schwarz $\Rightarrow$

$$((y+z)+(z+x)+(x+y))\left(\sum_{cyc}\frac{x^2}{y+z}\right)\geq(x+y+z)^2$$

$$\Longleftrightarrow \quad \sum_{cyc}\frac{x^2}{y+z}\geq\frac{1}{2}(x+y+z).$$

原式 $\Leftarrow x+y+z \geq 3 \Leftarrow x+y+z \geq 3\cdot(xyz)^{1/3} = 3.$ $\square$

<u>类似变形</u> (韩国, 1998) $x, y, z > 0$, $x+y+z = xyz$.

求证: $\sum_{cyc} \frac{1}{\sqrt{1+x^2}} \leq \frac{3}{2}$

<u>解答</u> $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$ $\Rightarrow ab+bc+ca = 1$.

$$\text{原式} \Leftrightarrow \sum_{cyc} \frac{a}{\sqrt{1+a^2}} \leq \frac{3}{2}$$

$$\Leftrightarrow \sum_{cyc} \frac{a}{\sqrt{a^2+ab+bc+ca}} \leq \frac{3}{2} \qquad (\text{齐次化})$$

$$\Leftrightarrow \sum_{cyc} \frac{a}{\sqrt{(a+b)(a+c)}} \leq \frac{3}{2}$$

均值 $\Rightarrow$
$$\frac{a}{\sqrt{(a+b)(a+c)}} = \frac{a}{(a+b)(a+c)} \cdot \sqrt{(a+b)(a+c)}$$
$$\leq \frac{a}{(a+b)(a+c)} \cdot \frac{1}{2}(a+b+a+c)$$
$$= \frac{1}{2}\left(\frac{a}{a+c} + \frac{a}{a+b}\right)$$

$$\Rightarrow \text{LHS} \leq \frac{1}{2}\sum_{cyc}\left(\frac{a}{a+c} + \frac{a}{a+b}\right) = \frac{3}{2}. \quad \square$$

以下是一个经典结论.

<u>定理1</u> $a, b, c > 0$, 则 $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.

<u>证法一</u> $x = b+c$, $y = c+a$, $z = a+b$ (注: 这不是Ravi代换)

$$\text{原式} \Leftrightarrow \sum_{cyc} \frac{y+z-x}{2x} \geq \frac{3}{2} \Leftrightarrow \sum_{cyc} \frac{y+z}{x} \geq 6.$$

但 $\text{LHS} = \frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{z}$

$$\geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{1/6} = 6. \quad \square$$

<u>证法二</u> $x = \frac{a}{b+c}$, $y = \frac{b}{a+c}$, $z = \frac{c}{a+b}$.

$$\Rightarrow \sum_{cyc} f(x) = \sum_{cyc} \frac{a}{a+b+c} = 1, \quad f(t) = \frac{t}{1+t}.$$

注意 $f(t)$ 在 $t \in (0, \infty)$ 凹.

Jensen $\Rightarrow$ $f\left(\frac{x+y+z}{3}\right) \geq \frac{1}{3}\sum_{cyc} f(x) = \frac{1}{3} = f\left(\frac{1}{2}\right)$.

而 f 单调增

$\Rightarrow$ $x+y+z \geq \frac{3}{2}$. $\square$

证法三 与上述类似，待证 $T = \frac{1}{3}(x+y+z) \geq \frac{1}{2}$，$\sum_{cyc} \frac{x}{1+x} = 1$.

而 $\sum_{cyc} \frac{x}{1+x} = 1 \Leftrightarrow \sum_{cyc} x(1+y)(1+z) = \prod_{cyc}(1+x)$

$\Leftrightarrow \sum_{cyc} xyz + x + xy + xz = 1 + x + y + z + \left(\sum_{cyc} xy\right) + xyz$

$\Leftrightarrow 2xyz + xy + yz + zx = 1.$

均值 $\Rightarrow$ $1 = 2xyz + xy + yz + zx \leq 2T^3 + 3T^2$

$\Rightarrow 2T^3 + 3T^2 - 1 \geq 0$

$\Rightarrow (2T-1)(T+1)^2 \geq 0$

$\Rightarrow T \geq \frac{1}{2}$. $\square$

Tips 这类齐次式的上界：直接将每个变量替换为它们的算术平均值.

例3 (IMO 2000, P2) $a, b, c > 0$，$abc = 1$，

证明：$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1$.

解答 (Ilan Vardi) $abc = 1$，可设 $a \geq 1 \geq b$.

有 $1-\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)$

$= \left(c+\frac{1}{c}-2\right)\left(a+\frac{1}{b}-1\right) + \frac{(a-1)(1-b)}{a}$.

注 本质上是逐步调整法，记 $f(a,b,c) = \text{LHS}$.

么 先证 $f(a,b,c) \leq f(a,b,1)$ （利用 $abc=1$ 去掉 c）

后证 $f(a,b,1) \leq f(1,1,1) = 1$.

<u>另解1</u> $a=\frac{x}{y},\ b=\frac{y}{z},\ c=\frac{z}{x}$.

原式 $\Longleftrightarrow \prod_{cyc}\frac{x+z-y}{y}\le 1 \Longleftrightarrow \prod_{cyc}(x+z-y)\le xyz$

↑ Ravi中的重要不等式.

$(\Longleftrightarrow R\geq 2r \Longleftrightarrow \sum_{cyc}\cos A\le \frac{3}{2})$

另给一个检验方法：设 $z\geq y\geq x$,

$y-x=p\geq 0,\quad z-x=q\geq 0.$

则 $xyz-\prod_{cyc}(x+y-z)=\underbrace{(p^2-pq+q^2)}_{\geq (p-q)^2\geq 0}x+\underbrace{(p^3+q^3-p^2q-pq^2)}_{=(p-q)^2(p+q)\geq 0}$. □

<u>另解2</u> (IMO Shortlist) $abc=1$

$$\Rightarrow\quad 2=\frac{1}{a}\left(a-1+\frac{1}{b}\right)+c\left(b-1+\frac{1}{c}\right)$$

$$2=\frac{1}{b}\left(b-1+\frac{1}{c}\right)+a\left(c-1+\frac{1}{a}\right)$$

$$2=\frac{1}{c}\left(c-1+\frac{1}{a}\right)+b\left(a-1+\frac{1}{b}\right)$$

设 $u=a-1+\frac{1}{b},\ v=b-1+\frac{1}{c},\ w=c-1+\frac{1}{a}$.

$\Rightarrow$ 至多 u,v,w 一者为负.

若有负，则 $uvw<0<1$. 设 $u,v,w\geq 0$.

$\Rightarrow\quad 2=\frac{1}{a}u+cv\geq 2\sqrt{\frac{c}{a}uv}\Rightarrow uv\le\frac{a}{c}$

类似有 $vw\le\frac{b}{a},\ wu\le\frac{c}{b}$.

$\Rightarrow\quad (uvw)^2\le 1,\quad u,v,w\geq 0.$ □

<u>例4</u> $a,b,c>0$. $a+b+c=1$. 求证:

$$\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{\sqrt{abc}}{c+ab}\le 1+\frac{3\sqrt{3}}{4}.$$

<u>解答</u> 原式 $\Longleftrightarrow \frac{1}{1+\frac{bc}{a}}+\frac{1}{1+\frac{ca}{b}}+\frac{\sqrt{\frac{ab}{c}}}{1+\frac{ab}{c}}\le 1+\frac{3\sqrt{3}}{4}$

设 $x=\sqrt{\frac{bc}{a}},\ y=\sqrt{\frac{ca}{b}},\ z=\sqrt{\frac{ab}{c}}$.

$\Leftrightarrow$ $\dfrac{1}{1+x^2}+\dfrac{1}{1+y^2}+\dfrac{z}{1+z^2}\le 1+\dfrac{3\sqrt{3}}{4}$，

$x, y, z>0$， $xy+yz+zx=1$.

则 $\exists A, B, C\in(0,\pi)$， $A+B+C=\pi$

s.t. $x=\tan\dfrac{A}{2}$，$y=\tan\dfrac{B}{2}$，$z=\tan\dfrac{C}{2}$.

原式 $\Leftrightarrow$ $\dfrac{1}{1+\tan^2\frac{A}{2}}+\dfrac{1}{1+\tan^2\frac{B}{2}}+\dfrac{\tan\frac{C}{2}}{1+\tan^2\frac{C}{2}}\le 1+\dfrac{3\sqrt{3}}{4}$

$\Leftrightarrow$ $1+\dfrac{1}{2}(\cos A+\cos B+\sin C)\le 1+\dfrac{3\sqrt{3}}{4}$

$\Leftrightarrow$ $\cos A+\cos B+\sin C\le\dfrac{3\sqrt{3}}{2}$

可以用调整方法，或：

$\cos A+\cos B=2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$， $\left|\dfrac{A-B}{2}\right|<\dfrac{\pi}{2}$

$\Rightarrow$ $\cos A+\cos B\le 2\cos\dfrac{A+B}{2}=2\cos\left(\dfrac{\pi-C}{2}\right)$

原式 $\Leftrightarrow$ $2\cos\dfrac{\pi-C}{2}+\sin C=2\sin\dfrac{C}{2}+\underset{\substack{\uparrow\\ C\in(0,\pi)}}{\sin C}\le\dfrac{3\sqrt{3}}{2}$ (练习). □

例5 (伊朗, 1998) $x, y, z>1$， $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=2$.

求证：$\sqrt{x+y+z}\ge\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$.

解答 $a=\sqrt{x-1}$，$b=\sqrt{y-1}$，$c=\sqrt{z-1}$.

条件 $\Leftrightarrow$ $\dfrac{1}{a^2+1}+\dfrac{1}{b^2+1}+\dfrac{1}{c^2+1}=2$

$\Leftrightarrow$ $a^2b^2+b^2c^2+c^2a^2+2a^2b^2c^2=1$

原式 $\Leftrightarrow$ $\sqrt{a^2+b^2+c^2+3}\ge a+b+c$

$\Leftrightarrow$ $ab+bc+ac\le\dfrac{3}{2}$.

令 $p=bc$，$q=ac$，$r=ab$.

$\Rightarrow$ $p^2+q^2+r^2+2pqr=1$

$\Rightarrow$ $\exists A, B, C\in(0,\frac{\pi}{2})$， $A+B+C=\pi$， s.t. $p=\cos A$，$q=\cos B$，$r=\cos C$.

往证 $p+q+r\le\frac{3}{2} \iff \cos A+\cos B+\cos C\le\frac{3}{2}$. $\square$

(在 1/4 周期可以直接用 Jensen).

例6 (IMO Shortlist, 2001) $x_1,\cdots,x_n\in\mathbb{R}$. 求证:

$$\frac{x_1}{1+x_1^2}+\frac{x_2}{1+x_1^2+x_2^2}+\cdots+\frac{x_n}{1+x_1^2+\cdots+x_n^2}<\sqrt{n}.$$

解答一 只看 $x_1,\cdots,x_n\geqslant 0$ 即可. 设 $x_0=1$.

$$y_i=x_0^2+\cdots+x_i^2,\quad x_i=\sqrt{y_i-y_{i-1}}.$$

原式 $\iff \displaystyle\sum_{i=1}^n\frac{\sqrt{y_i-y_{i-1}}}{y_i}<\sqrt{n}$, $y_i\geqslant y_{i-1}$.

左式分母 $y_i\geqslant\sqrt{y_i\cdot y_{i-1}}$

$$\Rightarrow \sum_{i=1}^n\frac{\sqrt{y_i-y_{i-1}}}{y_i}\le\sum_{i=1}^n\frac{\sqrt{y_i-y_{i-1}}}{\sqrt{y_iy_{i-1}}}=\sum_{i=1}^n\sqrt{\frac{1}{y_{i-1}}-\frac{1}{y_i}}.$$

Cauchy-Schwarz $\Rightarrow \displaystyle\sum_{i=1}^n\sqrt{\frac{1}{y_{i-1}}-\frac{1}{y_i}}\le\sqrt{n\sum_{i=1}^n\left(\frac{1}{y_{i-1}}-\frac{1}{y_i}\right)}$

$$=\sqrt{n\left(\frac{1}{y_0}-\frac{1}{y_n}\right)}<\sqrt{n}$$

$(y_0=1,\ y_n>0)$. $\square$

解答二 设 $x_0=0$, $x_1,\cdots,x_n\geqslant 0$. 取

$$t_i=\frac{x_i}{\sqrt{x_0^2+\cdots+x_i^2}},\quad c_i=\frac{1}{\sqrt{1+t_i^2}},\quad s_i=\frac{t_i}{\sqrt{1+t_i^2}}.$$

"tan" "cos" "sin".

$$\Rightarrow \frac{x_i}{x_1^2+\cdots+x_i^2}=c_0\cdots c_is_i,\quad s_i=\sqrt{1-c_i^2}.$$

原式 $\iff c_0c_1\sqrt{1-c_1^2}+c_0c_1c_2\sqrt{1-c_2^2}+\cdots+c_0c_1\cdots c_n\sqrt{1-c_n^2}<\sqrt{n}$.

而 $0\le c_i\le 1$, 有

$$\sum_{i=1}^n c_0\cdots c_i\sqrt{1-c_i^2}\le\sum_{i=1}^n c_0\cdots c_{i-1}\sqrt{1-c_i^2}$$

$$=\sum_{i=1}^n\sqrt{(c_0\cdots c_{i-1})^2-(c_0\cdots c_i)^2}$$

Cauchy-Schwarz $\Rightarrow \le\sqrt{n\displaystyle\sum_{i=1}^n(c_0\cdots c_{i-1})^2-(c_0\cdots c_i)^2}=\sqrt{n(1-(c_0\cdots c_n)^2)}$. $\square$