构造上下界

对于此前出现的经典不等式原型，我们给出新证明.

定理 (Nesbitt) $a, b, c > 0$，则有

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

证明一 $\left(\frac{a}{b+c} - \frac{1}{2}\right)^2 \geq 0$，

$$\Rightarrow \frac{a}{b+c} \geq \frac{1}{4} \cdot \frac{8\cdot\left(\frac{a}{b+c}\right) - 1}{\left(\frac{a}{b+c}\right) + 1} = \frac{8a-b-c}{4(a+b+c)}.$$

$$\Rightarrow \sum_{cyc} \frac{a}{b+c} \geq \sum_{cyc} \frac{8a-b-c}{4(a+b+c)} = \frac{3}{2}. \qquad \square$$

证明二 断言 $\frac{a}{b+c} \geq \frac{3a^{3/2}}{2(a^{3/2}+b^{3/2}+c^{3/2})} \Leftrightarrow 2(a^{3/2}+b^{3/2}+c^{3/2}) \geq 3a^{\frac{1}{2}}(b+c)$

均值 $\Rightarrow$ $a^{3/2}+b^{3/2}+b^{3/2} \geq 3a^{\frac{1}{2}}b$

$a^{3/2}+c^{3/2}+c^{3/2} \geq 3a^{\frac{1}{2}}c$

$\Rightarrow 2(a^{3/2}+b^{3/2}+c^{3/2}) \geq 3a^{\frac{1}{2}}(b+c)$.

所以 $$\sum_{cyc}\frac{a}{b+c} \geq \sum_{cyc}\frac{3a^{3/2}}{2(a^{3/2}+b^{3/2}+c^{3/2})} = \frac{3}{2}. \qquad \square$$

下面是一般方法：设目标为 $\sum_{cyc} F(x,y,z) \geq C$.

若 $G(x,y,z)$ 满足：(1) $F(x,y,z) \geq G(x,y,z)$，$\forall x,y,z>0$

(2) $\sum_{cyc} G(x,y,z) = C$，$\forall x,y,z>0$，

则 $\sum_{cyc} F(x,y,z) \geq \sum_{cyc} G(x,y,z) = C$.

E.g. $F(x,y,z) \geq \frac{x}{x+y+z}$，$\forall x,y,z>0$

$\Rightarrow \sum_{cyc} F(x,y,z) \geq 1$.

注 G的选取可以是多样的.

例1 a,b,c 是 $\triangle ABC$ 边长. 证明:

$$\sum_{cyc}\frac{a}{b+c}<2.$$

解答 (不用Ravi代换)

由三角不等式 $\Rightarrow \sum_{cyc}\frac{a}{b+c}<\sum_{cyc}\frac{a}{\frac{1}{2}(a+b+c)}=2.$ □

例2 a,b,c>0. 证明:

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\ge 1$$

讨论 本题动机如下:

欲求 $(x+y+z)^2$ 下界, $x,y,z>0$.

常见: $\ge 3(xy+yz+zx)$, $\ge 9(xyz)^{\frac{2}{3}}$, etc.

打破对称性: $(x+y+z)^2=x^2+\underbrace{y^2+z^2+xy+xy+yz+yz+zx+zx}_{\ge 8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}}}$.

$$\Rightarrow (x+y+z)^2\ge x^2+8x^{1/2}y^{3/4}z^{3/4}$$
$$=x^{1/2}(x^{3/2}+8y^{3/4}z^{3/4}).$$

解答 已知 $\sum_{cyc}\frac{x^{3/4}}{\sqrt{x^{3/2}+8y^{3/4}z^{3/4}}}\ge\sum_{cyc}\frac{x}{x+y+z}=1$

代入 $a=x^{3/4}$, $b=y^{3/4}$, $c=z^{3/4}$

$\rightsquigarrow$ 上式 $\Leftrightarrow \sum_{cyc}\frac{a}{\sqrt{a^2+8bc}}\ge 1.$ □

例3 (IMO 2005, P3) $x,y,z>0$. $xyz\ge 1$, 求证:

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\ge 0.$$

解答一 原式 $\Leftrightarrow \sum_{cyc}\left(\frac{x^2-x^5}{x^5+y^2+z^2}+1\right)\le 3.$

$$\Leftrightarrow \sum_{cyc}\frac{x^2+y^2+z^2}{x^5+y^2+z^2}\le 3.$$

$xyz \geq 1$，Cauchy-Schwarz：

$$\Rightarrow (x^5+y^2+z^2)(yz+y^2+z^2) \geq (x^2+y^2+z^2)^2$$

$$\Rightarrow \sum_{cyc} \frac{x^2+y^2+z^2}{x^5+y^2+z^2} \leq \sum_{cyc} \frac{yz+y^2+z^2}{x^2+y^2+z^2}$$

则希望 $\sum_{cyc}(yz+y^2+z^2) = 2(x^2+y^2+z^2)+(xy+yz+zx) \leq 3(x^2+y^2+z^2)$

$$\Leftrightarrow xy+yz+zx \leq x^2+y^2+z^2 \quad \text{(均值/排序)}. \qquad \square$$

<u>解答二</u>：主要思想：$\sum_{cyc} \frac{x^5}{x^5+y^2+z^2} \geq 1 \geq \sum_{cyc} \frac{x^2}{x^5+y^2+z^2}$.

先证左式：

$$y^4+z^4 \geq y^3z+yz^3 = yz(y^2+z^2)$$

$$\Rightarrow x(y^4+z^4) \geq xyz(y^2+z^2) \geq y^2+z^2.$$

$$\Leftrightarrow \sum_{cyc} \frac{x^5}{x^5+y^2+z^2} \geq \sum_{cyc} \frac{x^5}{x^5+x(y^4+z^4)} = \sum_{cyc} \frac{x^4}{x^4+y^4+z^4} = 1.$$

还需证右式，有两种方法：

(法一) $xyz \geq 1$，Cauchy-Schwarz：

$$\Rightarrow (x^5+y^2+z^2)(yz+y^2+z^2) \geq (x^2+y^2+z^2)^2$$

$$\Leftrightarrow \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2} \geq \frac{x^2}{x^5+y^2+z^2}$$

$$\Rightarrow \sum_{cyc} \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2} \geq \sum_{cyc} \frac{x^2}{x^5+y^2+z^2}.$$

希望 $1 \geq \sum_{cyc} \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2} \Leftrightarrow (x^2+y^2+z^2)^2 \geq 2\sum_{cyc} x^2y^2 + \sum_{cyc} x^2yz$

$$\Leftrightarrow \sum_{cyc} x^4 \geq \sum_{cyc} x^2yz \quad \text{(排序)}.$$

或：$\sum_{cyc} x^4 = \sum_{cyc} \frac{x^4+y^4}{2} \geq \sum_{cyc} x^2y^2 = \sum_{cyc} x^2\left(\frac{y^2+z^2}{2}\right) \geq \sum_{cyc} x^2yz$.

(法二) 断言：

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

$$\Leftarrow \quad \text{LHS} \geq \frac{x^2yz}{x^4+y^3z+yz^3} \qquad (*)$$

原因: $xyz \geq 1 \Rightarrow \dfrac{x^2yz}{x^4+y^3z+yz^3} = \dfrac{x^2}{\frac{x^5}{xyz}+y^2+z^2} \geq \dfrac{x^2}{x^5+y^2+z^2}$.

而 $(*) \Longleftrightarrow (2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3)$

$$\geq 4x^2yz(x^2+y^2+z^2)^2 \qquad (\text{齐次})$$

然而这是直接的:

$$\begin{aligned}
&(2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) - 4x^2yz(x^2+y^2+z^2)^2 \\
&= (x^8+x^4y^4+x^6y^2+x^6y^2+y^7z+y^3z^5) \\
&\quad + (x^8+x^4z^4+x^6z^2+x^6z^2+yz^7+y^5z^3) \\
&\quad + 2(x^6y^2+x^6z^2) - 6x^4y^3z - 6x^4yz^3 - 2x^6yz \\
&\geq 6\sqrt[6]{x^8\cdot x^4y^4\cdot x^6y^2\cdot x^6y^2\cdot y^7z\cdot y^3z^5} + 6\sqrt[6]{x^8\cdot x^4z^4\cdot x^6z^2\cdot x^6z^2\cdot yz^7\cdot y^5z^3} \\
&\quad + 2\sqrt{x^6y^2\cdot x^6z^2} - 6x^4y^3z - 6x^4yz^3 - 2x^6yz \\
&= 0.
\end{aligned}$$

$$\Rightarrow \quad 1 = \sum_{cyc}\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \sum_{cyc}\frac{x^2}{x^5+y^2+z^2}. \qquad \square$$

解答三 (Iurie Boreico, 摩尔多瓦代表队特别奖)

有 $$\frac{x^5-x^2}{x^5+y^2+z^2} - \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{(x^3-1)^2x^2(y^2+z^2)}{x^3(x^2+y^2+z^2)(x^5+y^2+z^2)} \geq 0$$

而 $$xyz \geq 1 \Rightarrow \sum_{cyc}\frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{1}{x^2+y^2+z^2}\sum_{cyc}\left(x^2-\frac{1}{x}\right)$$

$$\geq \frac{1}{x^2+y^2+z^2}\sum_{cyc}(x^2-yz) \geq 0. \qquad \square$$

练习 (1) (USAMO夏令营, 2002) $a,b,c>0$. 求证

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3$$

(2) (APMO, 2005) $a,b,c>0$, $abc=8$. 求证

$$\sum_{cyc}\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} \geq \frac{4}{3}.$$