Jensen不等式

命题 $f:[a,b]\to\mathbb{R}$ 连续. 下列描述等价:

(1) $\forall n\in\mathbb{N}$, $x_1,\cdots,x_n\in[a,b]$, $\omega_1,\cdots,\omega_n>0$, $\omega_1+\cdots+\omega_n=1$,

$$\omega_1 f(x_1)+\cdots+\omega_n f(x_n)\geq f(\omega_1x_1+\cdots+\omega_nx_n).$$

(2) $\forall n\in\mathbb{N}$, $x_1,\cdots,x_n\in[a,b]$, $r_1,\cdots,r_n\in\mathbb{Q}_{>0}$, $r_1+\cdots+r_n=1$,

$$r_1f(x_1)+\cdots+r_nf(x_n)\geq f(r_1x_1+\cdots+r_nx_n).$$

(3) $\forall N\in\mathbb{N}$, $y_1,\cdots,y_N\in[a,b]$,

$$\frac{1}{N}(f(y_1)+\cdots+f(y_N))\geq f\left(\frac{y_1+\cdots+y_N}{N}\right)$$

(4) $\forall k\in\mathbb{N}\cup\{0\}$, $y_1,\cdots,y_{2^k}\in[a,b]$,

$$\frac{1}{2^k}\left(f(y_1)+\cdots+f(y_{2^k})\right)\geq f\left(\frac{y_1+\cdots+y_{2^k}}{2^k}\right).$$

(5) $\forall x,y\in[a,b]$,

$$\frac{1}{2}f(x)+\frac{1}{2}f(y)\geq f\left(\frac{x+y}{2}\right).$$

(6) $\forall\lambda\in(0,1)$, $x,y\in[a,b]$,

$$\lambda f(x)+(1-\lambda)f(y)\geq f(\lambda x+(1-\lambda)y).$$

证明 (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) 显然.

(2) $\Rightarrow$ (1): $\exists\{r_k(1)\},\cdots,\{r_k(n)\}$ s.t.

$$r_k(j)\to\omega_j,\ k\to\infty\quad(1\leq j\leq n)$$

且 $r_k(1)+\cdots+r_k(n)=1$, $k\in\mathbb{N}$.

(2) $\Rightarrow$ $r_k(1)f(x_1)+\cdots+r_k(n)f(x_n)\geq f(r_k(1)x_1+\cdots+r_k(n)x_n)$.

f连续 $\Rightarrow$ $\lim f(r_k(j))=f(\lim r_k(j))$

$$\Rightarrow\ \omega_1f(x_1)+\cdots+\omega_nf(x_n)\geq f(\omega_1x_1+\cdots+\omega_nx_n).$$

(3) $\Rightarrow$ (2): $\exists N\in\mathbb{N}$ s.t. $Nr_1,\cdots,Nr_n\in\mathbb{N}$.

$\leadsto$ $r_i=\dfrac{p_i}{N}$, $r_1+\cdots+r_n=1$. $p_1+\cdots+p_n=N$.

$(3) \Rightarrow r_1 f(x_1) + \cdots + r_n f(x_n)$

$$= \frac{1}{N}\Big(\underbrace{f(x_1) + \cdots + f(x_1)}_{p_1\text{项}} + \cdots + \underbrace{f(x_n) + \cdots + f(x_n)}_{p_n\text{项}}\Big)$$

$$\geq f\Big(\frac{1}{N}(p_1 x_1 + \cdots + p_n x_n)\Big).$$

$$= f(r_1 x_1 + \cdots + r_n x_n).$$

<u>(4) $\Rightarrow$ (3)</u> $k \gg 0$ s.t. $2^k > N$. 取 $a = \frac{1}{N}(y_1 + \cdots + y_N)$.

$(4) \Rightarrow \frac{1}{2^k}\big(f(y_1) + \cdots + f(y_N) + (2^k - n) f(a)\big)$

$$= \frac{1}{2^k}\Big(f(y_1) + \cdots + f(y_N) + \underbrace{f(a) + \cdots + f(a)}_{2^k - N\text{项}}\Big)$$

$$\geq f\Big(\frac{1}{2^k}\big(y_1 + \cdots + y_N + (2^k - N)a\big)\Big) = f(a).$$

$$\Rightarrow f(y_1) + \cdots + f(y_N) \geq N f(a) = N f\Big(\frac{y_1 + \cdots + y_N}{N}\Big).$$

<u>(5) $\Rightarrow$ (4)</u> 对 k 归纳. $k = 0, 1, 2$ 显然.

设 (4) 对 $k \geq 2$ 成立, $y_1, \cdots, y_{2^{k+1}} \in [a,b]$.

归纳假设 $\Rightarrow f(y_1) + \cdots + f(y_{2^k}) + f(y_{2^k+1}) + \cdots + f(y_{2^{k+1}})$

$$\geq 2^k f\Big(\frac{1}{2^k}(y_1 + \cdots + y_{2^k})\Big) + 2^k f\Big(\frac{1}{2^k}(y_{2^k+1} + \cdots + y_{2^{k+1}})\Big)$$

$$= 2^{k+1} \cdot \frac{1}{2}\Big(f\Big(\frac{1}{2^k}(y_1 + \cdots + y_{2^k})\Big) + f\Big(\frac{1}{2^k}(y_{2^k+1} + \cdots + y_{2^{k+1}})\Big)\Big)$$

$$\geq 2^{k+1} \cdot f\left(\frac{\frac{y_1 + \cdots + y_{2^k}}{2^k} + \frac{y_{2^k+1} + \cdots + y_{2^{k+1}}}{2^k}}{2}\right)$$

$$= 2^{k+1} f\Big(\frac{1}{2^{k+1}}(y_1 + \cdots + y_{2^{k+1}})\Big).$$

$\leadsto$ (4) 对 $k+1$ 成立.

(1) $\Rightarrow$ (6) $\Rightarrow$ (5) 显然. $\square$

<u>定义</u> $f: [a,b] \to \mathbb{R}$ 称为凸的, 如果 $\forall x, y \in [a,b]$, $\lambda \in (0,1)$,

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y).$$

推论1 (Jensen) $f:[a,b]\to\mathbb{R}$ 连续, 凸. $\forall x_1,\cdots,x_n\in(a,b)$,

$$\frac{1}{n}(f(x_1)+\cdots+f(x_n))\geq f\left(\frac{x_1+\cdots+x_n}{n}\right).$$

推论2 (加权Jensen) $f:[a,b]\to\mathbb{R}$ 连续, 凸.

设 $\omega_1,\cdots,\omega_n\geq 0$, $\omega_1+\cdots+\omega_n=1$.

$\forall x_1,\cdots,x_n\in(a,b)$,

$$\omega_1 f(x_1)+\cdots+\omega_n f(x_n)\geq f(\omega_1 x_1+\cdots+\omega_n x_n).$$

推论3 (凸性判别法(一)) $f:[a,b]\to\mathbb{R}$ 连续.

设 $\forall x,y\in[a,b]$, $f(x)+f(y)\geq 2f\left(\frac{x+y}{2}\right)$.

则 f 在 $[a,b]$ 上凸.

练习 (凸性判别法(二)) $f:[a,b]\to\mathbb{R}$ 连续, 在 (a,b) 二次可导,

下列等价: (1) $f''(x)\geq 0$, $\forall x\in(a,b)$,

(2) f 在 (a,b) 上凸.

补注 在主命题中, $(5)\Rightarrow(4)\Rightarrow(3)\Rightarrow(2)$ 不使用 f 的连续性.

推论4 $f:[a,b]\to\mathbb{R}$ (不一定连续), 满足

$$\forall x,y\in[a,b],\quad f(x)+f(y)\geq 2f\left(\frac{x+y}{2}\right).$$

则 $\forall x_1,\cdots,x_n\in[a,b]$, $r_1,\cdots,r_n\in\mathbb{Q}_{>0}$, $r_1+\cdots+r_n=1$,

有 $$r_1 f(x_1)+\cdots+r_n f(x_n)\geq f(r_1 x_1+\cdots+r_n x_n).$$

最后, 我们给出一个加权Jensen的标准证明.

证明 归纳. 对 $n=1,2$ 显然. 设对 n 成立,

取 $x_1,\cdots,x_n,x_{n+1}\in[a,b]$, $\omega_1,\cdots,\omega_{n+1}>0$.

由 $\sum\limits_{i=1}^{n}\frac{\omega_i}{1-\omega_{n+1}}=1$, 有

$$
\begin{aligned}
&\omega_1 f(x_1)+\cdots+\omega_{n+1}f(x_{n+1})\\
&=(1-\omega_{n+1})\left(\frac{\omega_1}{1-\omega_{n+1}}f(x_1)+\cdots+\frac{\omega_n}{1-\omega_{n+1}}f(x_n)\right)+\omega_{n+1}f(x_{n+1})\\
&\geqslant(1-\omega_{n+1})f\Big(\sum_{i=1}^{n}\frac{\omega_i}{1-\omega_{n+1}}x_i\Big)+\omega_{n+1}f(x_{n+1})\\
&\geqslant f\left((1-\omega_{n+1})\sum_{i=1}^{n}\frac{\omega_i}{1-\omega_{n+1}}x_i+\omega_{n+1}x_{n+1}\right)\\
&=f(\omega_1x_1+\cdots+\omega_{n+1}x_{n+1}).
\end{aligned}
$$

$\square$