比较定理.

引入序关系：记 $(x_1,\cdots,x_n)\succ(y_1,\cdots,y_n)$，

如果 (1) $x_1\geq\cdots\geq x_n$， $y_1\geq\cdots\geq y_n$

(2) $\forall\, 1\leq k\leq n$， $x_1+\cdots+x_k\geq y_1+\cdots+y_k$.

(3) $x_1+\cdots+x_n=y_1+\cdots+y_n$.

定理 $f:[a,b]\to\mathbb{R}$ 凸. 设 $x_1,\cdots,x_n,y_1,\cdots,y_n\in[a,b]$，

且 $(x_1,\cdots,x_n)\succ(y_1,\cdots,y_n)$，

则 $f(x_1)+\cdots+f(x_n)\geq f(y_1)+\cdots+f(y_n)$.

评论 在 $\triangle ABC$ 中，可以此求 $\cos A+\cos B+\cos C$ 的下界（上界为 $\frac{3}{2}$）.

$-\cos x$ 在 $(0,\frac{\pi}{2})$ 凸， $(\frac{\pi}{2},\frac{\pi}{2},0)\succ(A,B,C)$

$$\Rightarrow \cos A+\cos B+\cos C\geq\cos\frac{\pi}{2}+\cos\frac{\pi}{2}+\cos 0=1.$$

同理，$\tan^2(\frac{x}{4})$ 在 $[0,\pi]$ 凸， $(\frac{\pi}{4},0,0)\succ(\frac{A}{4},\frac{B}{4},\frac{C}{4})$.

$$\Rightarrow 21-12\sqrt{3}=3\tan^2\left(\frac{\pi}{12}\right)\underbrace{\leq\tan^2\frac{A}{4}+\tan^2\frac{B}{4}+\tan^2\frac{C}{4}}_{\text{Jensen}}$$

比较定理 $$\leq\tan^2\frac{\pi}{4}+\tan^2 0+\tan^2 0=1.$$

例 (IMO 1999, P2) $n\geq 2$. 求最小的 C 使

$$\forall\, x_1,\cdots,x_n\geq 0,\quad \sum_{1\leq i<j\leq n}x_ix_j(x_i^2+x_j^2)\leq C\Big(\sum_{1\leq i\leq n}x_i\Big)^4.$$

解答 可正则化：$x_1+\cdots+x_n=1$.

$$\leadsto\quad \sum_{1\leq i<j\leq n}x_ix_j(x_i^2+x_j^2)=\sum_{i=1}^n f(x_i),$$

$$\left(f(x)=x^3-x^4,\quad 0\leq x\leq\frac{1}{2}\right).$$

不妨设 $x_1 \geq x_2 \geq \cdots \geq x_n$,

(1) 若 $\frac{1}{2} \geq x_1$, 则 $(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0) \succ (x_1, \cdots, x_n)$.

$$\Rightarrow \quad \sum_{i=1}^{n} f(x_i) \leq f(\tfrac{1}{2}) + f(\tfrac{1}{2}) + f(0) + \cdots + f(0) = \frac{1}{8}$$

(2) 若 $\frac{1}{2} \leq x_1$, $x_1 = \frac{1}{2} + \varepsilon$, $0 \leq \varepsilon \leq \frac{1}{2}$.

$$\Rightarrow \quad (1-x_1, \underbrace{0, \cdots, 0}_{n-2\text{项}}) \succ (x_2, \cdots, x_n)$$

$$\Rightarrow \quad \sum_{i=2}^{n} f(x_i) \leq f(1-x_1) + (n-2) f(0) = f(1-x_1)$$

$$\begin{aligned}
\Rightarrow \quad \sum_{i=1}^{n} f(x_i) &\leq f(x_1) + f(1-x_1) \\
&= x_1(1-x_1)(x_1^2 + (1-x_1)^2) \\
&= (\tfrac{1}{4} - \varepsilon^2)(\tfrac{1}{2} + 2\varepsilon^2) \\
&= 2(\tfrac{1}{16} - \varepsilon^4) \leq \frac{1}{8}. \qquad \square
\end{aligned}$$