Problems for Putnam Seminar

P 1. **Putnam 04A6** *Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that*

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy \right)^2 dx$$
$$\le \left(\int_0^1 \int_0^1 f(x,y)dx\,dy \right)^2 + \int_0^1 \int_0^1 (f(x,y))^2\,dx\,dy.$$

P 2. **Putnam 04B2** *Let m and n be positive integers. Show that*

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

P 3. **Putnam 03A2** *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that*

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \le [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

P 4. **Putnam 03A3** *Find the minimum value of*

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

P 5. **Putnam 03A4** *Suppose that a, b, c, A, B, C are real numbers, $a \ne 0$ and $A \ne 0$, such that*

$$|ax^2 + bx + c| \le |Ax^2 + Bx + C|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \le |B^2 - 4AC|.$$

P 6. **Putnam 03B6** *Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that*

$$\int_0^1 \int_0^1 |f(x) + f(y)|\,dx\,dy \ge \int_0^1 |f(x)|\,dx.$$

P 7. **Putnam 02B3** *Show that, for all integers $n > 1$,*

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

P 8. **Putnam 01A6** *Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?*

P 9. **Putnam 99A5** *Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then*

$$|p(0)| \le C \int_{-1}^1 |p(x)|\,dx.$$

P 10. **Putnam 99B4** *Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \le f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.*

P 11. **Putnam 98B4** *Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all integers $k \ge 0$,*

$$0 \le \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} \le 1.$$

P 12. **Putnam 98B1** *Find the minimum value of*

$$\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$

for $x > 0$.

P 13. **Putnam 96B2** *Show that for every positive integer* n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

P 14. **Putnam 96B3** *Given that* $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, *find, with proof, the largest possible value, as a function of* n *(with* $n \geq 2$*), of*

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

P 15. **Putnam 91B6** *Let* a *and* b *be positive numbers. Find the largest number* c, *in terms of* a *and* b, *such that*

$$a^xb^{1-x} \leq a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u}$$

for all u *with* $0 < |u| \leq c$ *and for all* x, $0 < x < 1$.

P 16. (**CMJ 416, Joanne Harris**) *For what real values of* c *is*

$$\frac{e^x+e^{-x}}{2} \leq e^{cx^2}.$$

for all real x*?*

P 17. (**CMJ420, Edward T. H. Wang**) *It is known [Daniel I. A. Cohen, Basic Techniques of Combinatorial Theory, p.56] and easy to show that* $2^n < \binom{2n}{n} < 2^{2n}$ *for all integers* $n > 1$. *Prove that the stronger inequalities*

$$\frac{2^{2n-1}}{\sqrt{n}} < \binom{2n}{n} < \frac{2^{2n}}{\sqrt{n}}$$

hold for all $n \geq 4$.

P 18. (**CMJ379, Mohammad K. Azarian**) *Let* x *be any real number. Prove that*

$$(1-\cos x)\left|\sum_{k=1}^{n}\sin(kx)\right|\left|\sum_{k=1}^{n}\cos(kx)\right| \leq 2.$$

P 19. (**CMJ392 Robert Jones**) *Prove that*

$$\left(1+\frac{1}{x^2}\right)\left(x\sin\frac{1}{x}\right) > 1 \text{ for } x \geq \frac{1}{\sqrt{5}}.$$

P 20. (**CMJ431 R. S. Luthar**) *Let* $0 < \phi < \theta < \frac{\pi}{2}$. *Prove that*

$$[(1+\tan^2\phi)(1+\sin^2\phi)]^{\csc^2\phi} < [(1+\tan^2\theta)(1+\sin^2\theta)]^{\csc^2\theta}.$$

P 21. (**CMJ451, Mohammad K. Azarian**) *Prove that*

$$\pi^{\sec^2\alpha}\cos^2\alpha + \pi^{\csc^2\alpha}\sin^2\alpha \geq \pi^2,$$

provided $0 < \alpha < \frac{\pi}{2}$.

P 22. (**CMJ446, Norman Schaumberger**) *If x, y, and z are the radian measures of the angles in a (non-degenerate) triangle, prove that*

$$\pi \sin \frac{3}{\pi} \geq x \sin \frac{1}{x} + y \sin \frac{1}{y} + z \sin \frac{1}{z}.$$

P 23. (**CMJ461, Alex Necochea**) *Let $0 < x < \frac{\pi}{2}$ and $0 < y < 1$. Prove that*

$$x - \arcsin y \leq \frac{\sqrt{1 - y^2} - \cos x}{y},$$

with equality holding if and only if $y = \sin x$.

P 24. (**CMJ485 Norman Schaumberger**) *Prove that*
(1) if $a \geq b > 1$ or $1 > a \geq b > 0$, then $a^{b^b} b^{a^a} \geq a^{b^a} b^{a^b}$; and
(2) if $a > 1 > b > 0$, then $a^{b^b} b^{a^a} \leq a^{b^a} b^{a^b}$.

P 25. (**CMJ524 Norman Schaumberger**) *Let a, b, and c be positive real numbers. Show that*

$$a^a b^b c^c \geq \left(\frac{a+b}{2}\right)^a \left(\frac{b+c}{2}\right)^b \left(\frac{c+a}{2}\right)^c \geq b^a c^b a^c.$$

P 26. (**CMJ567 H.-J. Seiffert**) *Show that for all ditinct positive real numbers x and y,*

$$\left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2 < \frac{x - y}{2 \sinh \frac{x-y}{x+y}} < \frac{x+y}{2}.$$

P 27. (**CMJ572, George Baloglou and Robert Underwood**) *Prove or disprove that for $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\cosh \theta \leq \frac{1}{\sqrt{1 - \tan^2 \theta}}$.*

P 28. (**CMJ603, Juan-Bosco Romero Marquez**) *Let a and b be distinct positive real numbers and let n be a positive integer. Prove that*

$$\frac{a+b}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}} \leq \sqrt[n]{\frac{a^n + b^n}{2}}.$$

P 29. (**MM^{5}904, Norman Schaumberger**) *For $x > 2$, prove that*

$$\ln\left(\frac{x}{x-1}\right) \leq \sum_{j=0}^{\infty} \frac{1}{x^{2j}} \leq \ln\left(\frac{x-1}{x-2}\right).$$

P 30. (**MM1590, Constantin P. Niculescu**) *For given a, $0 < a < \frac{\pi}{2}$, determine the minimum value of $\alpha \geq 0$ and the maximum value of $\beta \geq 0$ for which*

$$\left(\frac{x}{a}\right)^\alpha \leq \frac{\sin x}{\sin a} \leq \left(\frac{x}{a}\right)^\beta.$$

(This generalize the well-known inequality due to Jordan, which asserts that $\frac{2x}{\pi} \leq \sin x \leq 1$ on $[0, \frac{\pi}{2}]$.)

P 31. (**MM1597, Constantin P. Niculescu**) *For every $x, y \in \left(0, \sqrt{\frac{\pi}{2}}\right)$ with $x \neq y$, prove that*

$$\left(\ln \frac{1 - \sin xy}{1 + \sin xy}\right)^2 \geq \ln \frac{1 - \sin x^2}{1 + \sin x^2} \ln \frac{1 - \sin y^2}{1 + \sin y^2}.$$

P 32. (**MM1599, Ice B. Risteski**) *Given $\alpha > \beta > 0$ and $f(x) = x^\alpha (1-x)^\beta$. If $0 < a < b < 1$ and $f(a) = f(b)$, show that $f'(\alpha) < -f'(\beta)$.*

P 33. (**MM Q197, Norman Schaumberger**) *Prove that if* $b > a > 0$*, then* $\left(\frac{a}{b}\right)^a \geq \frac{e^a}{e^b} \geq \left(\frac{a}{b}\right)^b$.

P 34. (**MM1618, Michael Golomb**) *Prove that* $0 < x < \pi$,

$$x\frac{\pi - x}{\pi + x} < \sin x < \left(3 - \frac{x}{\pi}\right) x\frac{\pi - x}{\pi + x}.$$

P 35. (**MM1634, Constantin P. Niculescu**) *Find the smallest constant* $k > 0$ *such that*

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \leq k(a+b+c)$$

for every $a, b, c > 0$.

P 36. (**MM1233, Robert E. Shafer**) *Prove that if* $x > -1$ *and* $x \neq 0$*, then*

$$\frac{x^2}{1 + x + \frac{x^2}{2} - \frac{\frac{x^4}{120}}{1+x+\frac{31}{252}x^2}} < [\ln(1+x)]^2 < \frac{x^2}{1 + x + \frac{x^2}{2} - \frac{\frac{x^4}{240}}{1+x+\frac{1}{20}x^2}}.$$

P 37. (**MM1236, Mihaly Bencze**) *Let the functions* f *and* g *be defined by*

$$f(x) = \frac{\pi^2 x}{2\pi^2 + 8x^2} \quad \text{and} \quad g(x) = \frac{8x}{4\pi^2 + \pi x^2}$$

for all real x*. Prove that if* A*,* B*, and* C *are the angles of an acuted-angle triangle, and* R *is its circumradius then*

$$f(A) + f(B) + f(C) < \frac{a+b+c}{4R} < g(A) + g(B) + g(C).$$

P 38. (**MM1245, Fouad Nakhli**) *For each number* x *in open interval* $(1, e)$ *it is easy to show that there is a unique number* y *in* (e, ∞) *such that* $\frac{\ln y}{y} = \frac{\ln x}{x}$*. For such an* x *and* y*, show that* $x + y > x\ln y + y\ln x$.

P 39. (**MM Q725, S. Kung**) *Show that* $(\sin x)y \leq \sin(xy)$*, where* $0 < x < \pi$ *and* $0 < y < 1$.

P 40. (**MM Q771, Norman Schaumberger**) *Show that if* $0 < \theta < \frac{\pi}{2}$*, then* $\sin 2\theta \geq (\tan\theta)^{\cos 2\theta}$.