

MMP for generalized pairs

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MMP Classify alg vars / \mathbb{C}

(Hironaka, 1964) It suffices to classify sm alg vars.

MMP for surfaces X sm proj var

Can contract any (-1) -curve C on X

$$K_X \cdot C = -1, C^2 = -1.$$

$$\hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots \hookrightarrow X_{\min}.$$

Possibilities of X_{\min} :

(1) $X_{\min} \rightarrow \mathbb{Z}$ \mathbb{P}^1 -fibration or $X_{\min} = \mathbb{P}^2$.

(2) $K_{X_{\min}}$ is nef ($K \cdot C \geq 0$), the semiample case

$$X_{\min} \xrightarrow{|\mu \cdot K_{X_{\min}}|} \mathbb{Z}$$

(2.1) $\dim \mathbb{Z} = 2$, $K_{\mathbb{Z}}$ ample.

(2.2) $\dim \mathbb{Z} = 1$

(2.3) $\dim \mathbb{Z} = 0$, $K_{\mathbb{Z}}$ Calabi-Yau.

Higher-dim MMP

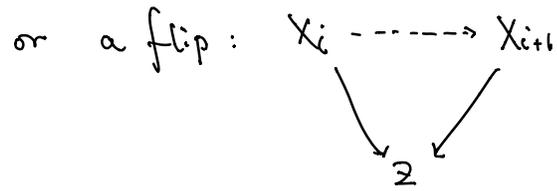
Issue Cannot define (-1) -curves.

\hookrightarrow Contract curves C instead s.t.

(1) $K_X \cdot C < 0$, (2) C is negative external ray in $\overline{NE}(X)$.

$$\hookrightarrow X \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_{\min}$$

s.t. $X_i \dashrightarrow X_{i+1}$ is either a divisorial contraction



s.t. (a) $X_i \rightarrow Z$ small, $\dim \text{Exc}(X_i \rightarrow Z) \leq \dim X - 2$.

(b) K_{X_i} is anti-ample / Z .

(c) $K_{X_{i+1}}$ is ample / Z .

As for X_{\min} :

(1) $X_{\min} \rightarrow Z$ MFS: $\dim X > \dim Z$, $\rho(X_{\min}/Z) = 1$.

$K_{X_{\min}}$ anti-ample.

(2) $K_{X_{\min}}$ nef (Abundance: $K_{X_{\min}}$ semiample

$X_{\min} \xrightarrow{mK_{X_{\min}}} Z$.)

What we know:

(1) Cone thm: $\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{\text{countable}} \mathbb{R}_+[C_i]$

(2) Contraction thm: May contract C_i (rational curve).

(3) Flip exists [BCHM 10].

(4) Flip termination: $\dim X \leq 3$.

(5) Abundance: $\dim X \leq 3$.

Canonical bundle formula

$X \rightarrow Z$, $K_X \sim_{\mathbb{Q}_Z} 0$.

Can write $K_X \sim a \cdot f^*(K_Z + B_Z + M_Z)$

where (i) $B_Z := \sum_{\mathbb{D}} (1 - \text{lc}_{\mathbb{D}}(X; f^*\mathbb{D})) \cdot \mathbb{D}$

log canonical cycle $\text{lc}_{\mathbb{Q}}(X; f^*D)$
 $:= \text{sup}\{t \mid (X, tf^*D) \text{ is lc over } \mathbb{Q}\}$

$$\left(\begin{array}{l} (X, B) \leftarrow \frac{\log f}{\text{res}} (Y, B_Y) \\ \text{s.t. } h^*(K_X + B) = K_Y + B_Y. \\ \text{lc condition means } B_Y \leq 1. \end{array} \right)$$

(ii) $M_Z: \exists g: Z' \rightarrow Z$ s.t. $M_{Z'}$ is nef & $M_Z = g_* M_{Z'}$

$$g^*(K_Z + B_Z + M_Z) = K_{Z'} + B_{Z'} + M_{Z'}. \quad X' \longrightarrow X$$

with $B_{Z'} = \sum_{\mathbb{Q}} (1 - \text{lc}(X', B'; f^*D)) \cdot D$.

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow \\ Z' & \xrightarrow{g} & Z \end{array}$$

Structure of $K_Z + B_Z + M_Z$

(1) [Ambro] M_Z nef & abundant

$$\rightarrow K_Z + B_Z + M_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z, \quad (Z, \Delta_Z) \text{ klt.}$$

(2) [PS09] Conj $M_{Z'}$ is semiample (too hard.)

(3) [BZ16] Directly study $(Z, B_Z + M_Z)$
 generalized pair

Sing of generalized pair:

$$(Z, B_Z + M_Z) \longleftarrow (Z', B_{Z'} + M_{Z'})$$

$$\cdot B_{Z'} \leq 1 : \text{lc} \quad , \quad \cdot B_{Z'} < 1 : \text{klt.}$$

What we know about a generalized pair $(X, B+M)$:

(1) Cone thm: [Hacon-Li 21] $\overline{NE}(X) = \overline{NE}(X)_{K_X + B + M \geq 0} + \sum R_i + [C_i]$.

(2) Contraction thm: [Xie 22] May contract C_i
 [Chen-Liu-Xie 23].

(3) Flip exists: [Liu-Xie 22]

(4) Flip termination: $\dim X \leq 3$ \leftarrow ample

(5) Abundance: failed; $(X, B+A+M)$,

$K_X + B + M$ ample \Rightarrow semiample.

[Liu-Xie 22], [Tschanke-Xie 23]

Cor (1) Generalized lc singularities are Du Bois [Liu-Xie 22].

(2) Kodaira vanishing holds for lc generalized pairs

[Chen-Liu-Xie 23].