MMP for generalized pairs

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<u>MMP</u> Classify alg vars / $\mathbb{C}$

(Haronalka, 1964) It suffices to classify sm alg vars.

<u>MMP for surfaces</u> X sm proj var

Can contract any (-1)-curve C on X

$$K_X C = -1, \ C^2 = -1.$$

$\rightsquigarrow X_1 \rightsquigarrow X_2 \rightsquigarrow \cdots \rightsquigarrow X_{min}$.

Possibilities of X_{min}:

(1) $X_{min} \longrightarrow Z$ $\mathbb{P}^1$-fibration or $X_{min} = \mathbb{P}^2$.

(2) $K_{X_{min}}$ is nef ($K \cdot C \geq 0$), the semiample case

$$X_{min} \xrightarrow{|m K_{X_{min}}|} Z$$

(2.1) $\dim Z = 2$, K_Z ample.

(2.2) $\dim Z = 1$

(2.3) $\dim Z = 0$, K_Z Calabi-Yau.

<u>Higher-dim'l MMP</u>

<u>Issue</u> Cannot define (-1)-curves.

$\rightsquigarrow$ Contract curves C instead s.t.

(1) $K_X \cdot C < 0$, (2) C is negative external ray in $\overline{NE}(X)$.

$\rightsquigarrow X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{min}$

s.t. $X_i \dashrightarrow X_{i+1}$ is either a divisoral contraction

or a flip:
$$\begin{tikzcd} X_i \arrow[rr, dashed] \arrow[dr] & & X_{i+1} \arrow[dl] \\ & Z & \end{tikzcd}$$

s.t. (a) $X_i \to Z$ small, $\dim \operatorname{Exc}(X_i \to Z) \leq \dim X - 2$.

(b) K_{X_i} is anti-ample $/Z$.

(c) $K_{X_{i+1}}$ is ample $/Z$.

As for X_{min}:

(1) $X_{min} \to Z$ MFS: $\dim X > \dim Z$, $\rho(X_{min}/Z) = 1$,

$K_{X_{min}}|_F$ anti-ample.

(2) $K_{X_{min}}$ nef (Abundance: $K_{X_{min}}$ semiample

$X_{min} \xrightarrow{|mK_{X_{min}}|} Z$.)

What we know:

(1) Cone thm: $\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{\text{Countable}} \mathbb{R}_+[C_i]$

(2) Contraction thm: May contract C_i (rational curve).

(3) Filp exists [BCHM 10].

(4) Filp termination: $\dim X \leq 3$.

(5) Abundance: $\dim X \leq 3$.

<u>Canonical bundle formula</u>

$X \to Z$, $K_X \sim_{\mathbb{Q}, Z} 0$.

Can write $K_X \sim a \cdot f^*(K_Z + B_Z + M_Z)$

where (i) $B_Z := \sum_D (1 - \operatorname{lct}_{\eta_D}(X; f^*D)) \cdot D$

log canonical cycle $\mathrm{lct}_{\eta_D}(X; f^*D)$

$$:= \sup\{t \mid (X, tf^*D) \text{ is lc over } \eta_D\}$$

$$\left(\begin{array}{l} (X,B) \xleftarrow[\text{res}]{\log h} (Y, B_Y) \\ \text{s.t. } h^*(K_X+B) = K_Y + B_Y. \\ \text{lc condition means } B_Y \le 1. \end{array} \right)$$

(ii) M_Z: $\exists\, g: Z' \to Z$ s.t. $M_{Z'}$ is nef & $M_Z = g_* M_{Z'}$

$$g^*(K_Z + B_Z + M_Z) = K_{Z'} + B_{Z'} + M_{Z'}.$$

with $B_{Z'} = \sum_D (1 - \mathrm{lct}(X', B'; f'^*D)) \cdot D.$

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d] \\ Z' \arrow[r, "g"] & Z \end{tikzcd}$$

Structure of $K_Z + B_Z + M_Z$

(1) [Ambro] M_Z nef & abundant

$$\Rightarrow K_Z + B_Z + M_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z, \quad (Z, \Delta_Z) \text{ klt}.$$

(2) [PS09] Conj $M_{Z'}$ is semiample (too hard.)

(3) [BZ16] Directly study $\underbrace{(Z, B_Z + M_Z)}_{\text{generalized pair}}$

Sing of generalized pair:

$$(Z, B_Z + M_Z) \longleftarrow (Z', B_{Z'} + M_{Z'})$$

- $B_{Z'} \le 1$: lc, • $B_{Z'} < 1$: klt.

What we know about a generalized pair $(X, B+M)$:

(1) Cone thm: [Hacon-Li 21] $\overline{NE}(X) = \overline{NE}(X)_{K_X+B+M\ge 0} + \sum \mathbb{R}_+[C_i]$.

(2) Contraction thm: [Xie 22] May contract C_i

[Chen-Liu-Xie 23].

(3) Flip exists: [Liu-Xie 22]

(4) Flip termination: $\dim X \leq 3$

(5) Abundance: failed; $(X, B + \overset{\text{ample}}{A} + M)$,

$K_X + B + M$ ample $\Rightarrow$ semiample.

[Liu-Xie 22], [Tsakanikas-Xie 23]

Cor (1) Generalized lc singularities are Du Bois [Liu-Xie 22].

(2) Kodaira vanishing holds for lc generalized pairs [Chen-Liu-Xie 23].