

BANACH–COLMEZ SPACES
(LEARNING SEMINAR IN p -ADIC GEOMETRY, BEIJING, FALL 2022)

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Time: 2:00-3:30 pm on Mondays

Location: MCM 110

Seminar Website: http://shizhang.li/seminars/BC%20spaces_22%20Fall.html

Purpose: The goal of this series of talks is to learn the basics of finite-dimensional Banach Spaces¹ (aka. Banach–Colmez spaces). We will study Colmez’s original paper [Col02] in detail. The latter part of the seminar will be devoted to relevant recent works such as [LB18, CN17, Niz19] (*update*: we will spend the last three weeks studying [CN17]). We will decide which topics to cover on the course of the seminar. Note that we plan to focus more on the topics and results that are NOT covered in [FS21, Chap. II].

Schedule: The following is an outline and suggestion for each talk. Please feel free to reorganize the materials if you wish.

Mentoring: As part of professional development, student/postdoc speakers are required to discuss their detailed lecture plan with one of the organizers (KS) and get feedback in advance.

Lecture 1. *Introduction and organization.* (9/5, Koji)

Outline the materials and distribute the weekly lectures to volunteers.

Lecture 2. *Sympathetic algebras and $\widehat{C\{X\}}$.* (9/19, Shizhang)

Define sympathetic algebras [Col02, p. 356] and show that $\widehat{C\{X\}}$ is sympathetic. For this, review some of the definitions (e.g. $\text{Spec } \Lambda$) and properties in [Col02, §2] if necessary, and discuss [Col02, §3].

Lecture 3. *Sympathetic closure.* (9/26, Yongquan)

Prove the existence of a sympathetic closure [Col02, §5.2-5.5; Thm. 5.18, Prop. 5.20]. As a corollary, mention [Sch13, Prop. 4.8]. Finally, discuss [Col02, §5.6].

Lecture 4. *A funny field \mathfrak{C} .* (10/10, Qijun)

Following [Col02, §6.1-6.5], introduce the subsets $\mathfrak{C} \subset \widehat{\mathfrak{C}} \subset \widehat{C\{X\}}$ and show that they are rings. Note that [Col02, Prop. 6.8] uses [Col02, §4], which you need to explain briefly.

Lecture 5. *Vector Spaces.* (10/17, Yupeng)

First cover [Col02, §6.6-6.9]. Mention [Col02, Thm. 10.5] if you wish. Then explain Vector Spaces [Col02, §7.1, 7.2].

Date: September 6, 2022.

¹Capital S!

Lecture 6. *Finite-dimensional Banach Spaces.* (10/24, Zhefan)

Cover [Col02, §7.3-7.8]. This talk gives the definition and basic properties of finite-dimensional Banach Spaces.

Lecture 7. *p-adic period rings.* (10/31, Daxin)

Cover [Col02, §8]. The goal of this talk is to explain Fontaine's period rings in the framework of finite-dimensional Banach Spaces. .

Lecture 8. *Fundamental exact sequence.* (11/7, Yong-Suk)

Following [Col02, §9], prove that the sequence of Vector Spaces

$$0 \rightarrow E \rightarrow \mathbb{B}_{\max, E}^{\varphi_E=1} \rightarrow \mathbb{B}_{\mathrm{dR}} / \mathbb{B}_{\mathrm{dR}}^+ \rightarrow 0$$

is exact [Col02, Prop. 9.25]. Mention [Col02, Prop. 10.16] if you wish.

Lecture 9. *fa \implies a .* (11/14, Zekun)

Following [Col02, §11], prove that weak admissibility implies admissibility [Col02, Thm. 11.19].

Lecture 10. *Banach–Colmez spaces.* (11/21, Jiedong)

Explain [LB18, §2]. Then state [LB18, Thm 4.1] and prove some of the cases, using [LB18, §3] as a black box.

Lecture 11. *Pro-étale cohomology of affine spaces.* (11/28, Ruochuan)

Cover [LB18, §3]. If possible, comment on another proof [CN20].

Lecture 12. *The category \mathcal{BC} in terms of the Fargues–Fontaine curve.* (12/5, Heng)

Cover [LB18, §5-7]. Explain the key results and ideas clearly.

Lecture 13. *Overview and local syntomic computations.* (12/12, Koji)

Cover [CN17, §1-3]. Give a quick overview of the paper and introduce the period rings. Then explain local syntomic computations.

Lecture 14. *Syntomic cohomology and (φ, Γ) -modules.* (12/19, Shanwen)

Cover [CN17, §4]. The goal of this talk is to compare the local Fontaine–Messing period map and the Lazard type period map [CN17, Thm. 4.16]. Since the notation for the general case is too heavy, you may focus on $d = 1$.

Lecture 15. *Semistable comparison theorem.* (12/26)

Cover [CN17, §5]. Prove the semistable conjecture [CN17, Cor. 5.26] using the syntomic cohomology and Banach–Colmez spaces.

REFERENCES

- [CN17] Pierre Colmez and Wiesława Nizioł, *Syntomic complexes and p-adic nearby cycles*, *Inventiones mathematicae* **208** (2017), no. 1, 1–108.
- [CN20] ———, *On the cohomology of the affine space, p-adic Hodge theory*. Proceedings of the Simons symposium, Schloss Elmau, Germany, May 7–13, 2017, Cham: Springer, 2020, pp. 71–80 (English).
- [Col02] Pierre Colmez, *Espaces de Banach de dimension finie*, *J. Inst. Math. Jussieu* **1** (2002), no. 3, 331–439. MR 1956055
- [FS21] Laurent Fargues and Peter Scholze, *Geometrization of the local langlands correspondence*, 2021.
- [LB18] Arthur-César Le Bras, *Espaces de Banach–Colmez et faisceaux cohérents sur la courbe de Fargues–Fontaine*, *Duke Mathematical Journal* **167** (2018), no. 18, 3455 – 3532.
- [Niz19] Wiesława Nizioł, *Geometric syntomic cohomology and vector bundles on the Fargues–Fontaine curve*, *J. Algebr. Geom.* **28** (2019), no. 4, 605–648 (English).
- [Sch13] Peter Scholze, *p-adic Hodge theory for rigid-analytic varieties*, *Forum Math. Pi* **1** (2013), e1, 77. MR 3090230