$\mathcal{BC}$ in terms of FF curve. (after Le Bras).

Recall in Colmez paper, fix $C = \widehat{\overline{C}}$ in char 0 k residue field.

C perfectoid $C^\flat$ tilt.

Jiedong's talk:

define $\mathcal{BC}$ $(\widetilde{\mathcal{BC}})$ = smallest ab subcat of $\mathrm{Vect}_{\mathbb{Q}_p}(\mathrm{Perf}_{C,\text{proét}})$ $(\mathrm{Ab}(\mathrm{Perf}_{C,\text{proét}}))$.

containing $\underline{\mathbb{Q}_p}$, $\mathbb{G}_a$ and stable under extensions.

Goal of today's talk:

identify $\mathcal{BC}$ & $\widetilde{\mathcal{BC}}$ with subcat of $\mathbb{D} = D^b(\mathrm{Coh}(X))$.

X = absolute Fargues-Fontaine curve / $C^\flat$

§ Facts about (relative, adic) FF curves.

$S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid $\in \mathrm{Perf}_{C^\flat}$.

$Y_S = \mathrm{Spa}\, W(R^+) \setminus V(p[\varpi])$. $\varpi \in R^+$ a pseudo-unif.

Y_S = analytic adic space.

$$= \bigcup_{m,n\geq 1} \{ x \in Y_S \mid |\varpi|_x^n \leq |p|_x \leq |\varpi|_x^{\frac{1}{m}} \}.$$

$$=: \bigcup_{m,n\geq 1} Y_{S,m,n}$$ - rational open subsets of $\mathrm{Spa}\, W(R^+)$.

(affinoid Tate)

$B(S) := \mathcal{O}(Y_S)$.

$\varphi \circlearrowleft W(R^+)$ induces $\varphi \circlearrowleft Y_S$ free & properly discontinuous.

$X_S := Y_S / \varphi^{\mathbb{Z}}$ relative FF curve over S.

$X := X_{\mathrm{Spa}(C^\flat, \mathcal{O}_C^\flat)}$. absolute FF curve. studies by Fargues-Fontaine.

$B := B(\mathrm{Spa}(C^\flat, \mathcal{O}_C^\flat))$. was carefully studies. $B = B_{(0,1)}$ Koji's lecture

$P = \bigoplus_{d\geq 0} B^{\varphi = p^d}$ graded algebra / $B^{\varphi = \mathrm{Id}} = \mathbb{Q}_p$.

Fact: $X^{\mathrm{Sch}} := \mathrm{Proj}\, P$ is a 1-dim. noetherian regular scheme

$\exists\ X \longrightarrow X^{\mathrm{Sch}}$. morphism of loc. r'g spaces.

Thm (Kedlaya-Liu). $X \to X^{Sch}$ induces an equivalence of

$$\mathrm{Bun}_{X^{Sch}} \xrightarrow{\sim} \mathrm{Bun}_X.$$ and induces isomorphism

on cohomology gps of vector bundles (GAGA).

(D, φ_D) isocrystal $/k$. simple of slope $-\lambda$.

define $\mathcal{O}(\lambda)$ to be the v.b. associated with.

$$\bigoplus_{d \geq 0} (D \otimes_{W(k)[\frac{1}{p}]} B)^{\varphi = p^d} \curvearrowright \text{diagonal}.$$

$\rightsquigarrow$ $H^0(X, \mathcal{O}(\lambda)) = B^{\varphi^h = p^d}$ $(= 0 \quad \lambda < 0)$. $\lambda = \frac{d}{h}$, $(h, d) = 1$

$\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = H^0(X, \mathcal{O}(\mu - \lambda).)^{\oplus m.}$ $(= 0$ when $\lambda > \mu)$.

Thm (Fargues-Fontaine).

Every vector bundle $\mathcal{E}$ over X^{Sch}, $\exists$ noncan. isomorphic to.

$$\mathcal{E} \simeq \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}(\lambda)^{m_\lambda(\mathcal{E})}$$

Above thm $\Rightarrow$ given $\mathcal{E}$ $\rightsquigarrow$ multi set of $\lambda \in \mathbb{Q}$ with $m_\lambda(\mathcal{E}) \neq 0$

called the slopes of $\mathcal{E}$.

$\mathcal{E}$ is called semi-stable of slope λ if it only as slope λ.

$$\begin{array}{ccccccc} \mathrm{Isoc}_k^{\lambda} & \xleftrightarrow{\sim} & \mathrm{Bun}_X^{s.s.\lambda.} & \overset{\lambda=0}{\Rightarrow} & \mathrm{Vect}_{\mathbb{Q}_p} & \xleftrightarrow{\sim} & \mathrm{Bun}_X^{ss.0}. \\ D & \longrightarrow & \bigoplus_{d \geq 0} (D \otimes B)^{\varphi = p^d} & & V & \longmapsto & V \otimes \mathcal{O}_X. \\ & & & & H^0(X, \mathcal{E}). & \longleftarrow & \mathcal{E} \end{array}$$

Rmk: $\mathrm{Vect}_{\mathbb{Q}_p} \simeq \mathrm{Bun}_X^{ss.0}$ generalize to relative FF curve, with s.s. of 0.

replaced $\updownarrow$ by pointwise s.s. slope 0. by a thm of Kedlaya-Liu.

and: $H^0(X_S, V \otimes \mathcal{O}_{X_S}) \simeq \underline{V}(S)$.

$\swarrow$ set of closed pts of X^{Sch}

$|X^{Sch}| \longleftrightarrow$ (untilts of C^b in char 0) $/ \sim$.

$$x \longmapsto C_x := \kappa(x).$$

s.t. $\mathcal{O}^{sch}_{X,x} \simeq B^+_{dR}(C_x)$.

$\infty \longleftrightarrow C$ as untilt of C^b.

Rmk: Similar for X_S ; X_S is "the moduli of untilts of S"

$\mathcal{F} \in Coh(X^{sch})$ $\quad \mathcal{F} \simeq \mathcal{F}_{tor} \oplus \mathcal{F}_{free}$.

define slope of $\mathcal{F}_{tor}$ to be ∞.

§ $Coh(X)^-$ $\quad X = X^{sch}$.

$D = D^b(Coh(X))$.

$$Coh(X)^- = \left\{ \mathcal{F} \in D \;\middle|\; \begin{array}{l} H^i(\mathcal{F}) = 0 \text{ if } i \neq 0, -1. \\ H^0(\mathcal{F}) \text{ as slopes } \geq 0 \;,\; H^{-1}(\mathcal{F}) \text{ has slopes } < 0 \end{array} \right\}.$$

PROP: $Coh(X)^-$ is a heart of D ($\exists$ t-str).

Key is that $\mathrm{Hom}(\text{slope} \geq 0, \text{slope} < 0) = 0$.

$\rightsquigarrow$ $Coh(X)^-$ is an abelian category.

$\forall\, \mathcal{F} \in Coh(X)^-$

$$H^{-1}(\mathcal{F})[1] \longrightarrow \mathcal{F} \longrightarrow H^0(\mathcal{F}) \xrightarrow{+1}$$

is an exact triangle in D.

actually: $0 \to H^{-1}(\mathcal{F})[1] \to \mathcal{F} \to H^0(\mathcal{F}) \to 0$.

is exact in $Coh(X)^-$, and

$$\mathrm{Ext}^1_{Coh(X)^-}(H^0(\mathcal{F}), H^{-1}(\mathcal{F})[1])$$
$$= \mathrm{Ext}^1_D(H^0(\mathcal{F}), H^{-1}(\mathcal{F})[1]).$$
$$= \mathrm{Ext}^2_{Coh}(H^0(\mathcal{F}), H^{-1}(\mathcal{F})) = 0 \quad \text{since } X \text{ curve}.$$

$\rightsquigarrow$ $\mathcal{F} \simeq H^{-1}(\mathcal{F})[1] \oplus H^0(\mathcal{F})$.

$\mathcal{F} = (\mathcal{F}^{-1}, \mathcal{F}^0)$ $\qquad \mathcal{F}^{-1} = H^{-1}(\mathcal{F})$ slope <0

$\qquad \mathcal{F}^0 = H^0(\mathcal{F})$ slope ≥ 0.

Slogan. $\mathrm{Coh}(X)^-$ & $\mathrm{Coh}(X)$ has same objects. (up to isom).
but different morphism.

e.g. $\mathrm{Hom}_{\mathrm{Coh}(X)}((0, \mathcal{F}^0), (\mathcal{G}^{-1}, 0)) = 0$.

$\mathrm{Hom}_{\mathrm{Coh}(X)^-}((0, \mathcal{F}^0), (\mathcal{G}^{-1}, 0)) = \mathrm{Hom}_D(\mathcal{F}^0, \mathcal{G}^{-1}[1])$

$= \mathrm{Ext}^1_{\mathrm{Coh}(X)}(\mathcal{F}^0, \mathcal{G}^{-1}) \neq 0$

§ Main result.

Def-Lemma: $\exists$ well-defined functor.

$$T : \mathrm{Coh}(X)^- \longrightarrow \mathcal{BC} \longrightarrow \widetilde{\mathcal{BC}}$$

$$\mathcal{F} \simeq (\mathcal{F}^{-1}, \mathcal{F}^0) \longmapsto (S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{C^b, \mathrm{pro\acute{e}t}} \longmapsto H^0(X_S, \mathcal{F}^0_S) \oplus H^1(X_S, \mathcal{F}^{-1}_S)).$$

Pf: $S \longmapsto X_S$ is functorial.

$S \to \mathrm{Spa}\, C^b \longmapsto (X_S \longrightarrow X)$ $\qquad$ $\mathcal{F}_S^*$ the pull back of $\mathcal{F}^*$.

We will see. $H^i(X_S, \mathcal{F}_S)$ are $\mathbb{Q}_p$-v.s.

$\mathcal{F} \to \mathcal{G} \rightsquigarrow T(\mathcal{F}) \to T(\mathcal{G})$ automatic $\mathbb{Q}_p$-linear.

T induces $\mathrm{Coh}(X)^- \simeq \mathcal{BC} \simeq \widetilde{\mathcal{BC}}$

① $\mathcal{F} = \mathcal{O}_X \overset{T}{\longmapsto} (S \mapsto H^0(X_S, \mathcal{O}_{X_S})) = \underline{\mathbb{Q}_p}$.

②: $S \in \mathrm{Perf}_{C^b} \longleftrightarrow S^\# \in \mathrm{Perf}_C$.

$\iota : S^\# \longrightarrow X_S$

and $H^i(X_S, \iota_* \mathbb{B}^+_{\mathrm{dR}, S^\#}/t^k) \simeq H^i(S^\#, \mathbb{B}^+_{\mathrm{dR}, S^\#}/t^k)$.

in particular, $T(\iota_{\infty,*} C) = \mathbb{G}_a$. $\quad (\mathbb{B}^+_{\mathrm{dR}, S^\#}/t \simeq \mathcal{O}_{S^\#})$.

③. To check, ~~essential~~ image is inside $\mathcal{BC}$.

$$K_0(\mathrm{Coh}(X)^-) \overset{\sim}{=} K_0(\mathrm{Coh}(X)) \simeq K_0(\mathrm{Bun}_X).$$

$$[\mathcal{F}] \longmapsto [H^0(\mathcal{F})] - [H^1(\mathcal{F})]$$

$\wr$ rank ⊕ det

$$\mathbb{Z}[\mathcal{O}_X] \oplus \mathbb{Z}[\mathcal{O}_X(1)] \xrightarrow{\sim} \mathbb{Z}[\mathcal{O}_X] \oplus \mathrm{Pic}(X).$$

$\mathrm{Pic}(X) \simeq \mathbb{Z}$

$[\mathcal{O}(n)] \longleftarrow n$

Koji's lecture notes.

So it is enough to check.

$$T(\underset{\shortparallel}{\mathcal{O}_X}),\ T(\mathcal{O}_X(1)) \in \mathcal{BC}.$$

$H^0(S, \mathcal{O}_{X_S}) = = .$ $\underline{\mathbb{Q}_p}$

Lemma: Sympathetic C-algebras forms a basis for $\mathrm{Perf}_{C,\mathrm{pro\acute{e}t}}$.

Pf: Colmez' paper. "sympathetic closure". (Yongquan's talk). [KL1, Cor 5.2.12] □

$T(\mathcal{O}_X)(R) = B(R)^{\varphi = id} =$

$\downarrow$ $\qquad \mathbb{U}_1 := T(\mathcal{O}_X(1)).$

$\underline{\text{Will see}}$: $0 \to \underset{\substack{\shortparallel \\ \mathbb{Q}_p \ (R \text{ connected})}}{\underline{\mathbb{Q}_p}(R)} \to \mathbb{U}_1(R) = \underset{\substack{\shortparallel \\ B^+(R)^{\varphi=p}}}{B(R)}{}^{\varphi = p} \xrightarrow{\theta} \underset{\substack{\shortparallel \\ \mathbb{G}_a}}{\mathbb{V}^1_u(R)} \to 0$

$T(\mathcal{O}_X(1)) \in \mathcal{BC}$ $\quad \left(\bigcap_n \varphi^n (\mathbb{B}^+_{\max}(R)) \right)^{\varphi = p}$

$\shortparallel$

$\mathbb{B}^+_{\max}(R)^{\varphi = p}$

$\underline{\text{Thm}}$. T induces $\overset{\text{exact}}{}$ $\mathrm{Coh}(X)^- \simeq \mathcal{BC} \simeq \widetilde{\mathcal{BC}}$.

① T exact. $\mathrm{Coh}(X)^- \longrightarrow \widetilde{\mathcal{BC}}$

Assume ①. $\forall\ \mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)^-$

T induces

$$(*) \qquad \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^i(T(\mathcal{F}), T(\mathcal{G})).$$

$(*)$ is isom. $i = 0 \Rightarrow$ fully faithfulness

we know $T(\mathcal{O}_X) = \underline{\mathbb{Q}_p}$ $\quad T(i_{\infty *} C) \simeq \mathbb{G}_a$.

by definition of $\mathcal{BC}$. it is enough to show

$(*)$ is isom $i = 1$.

Lemma. $(*)$ is isom for $\forall\ \mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)^-$, $i = 0, 1$.

Lemma. $(*)$ is isom for. $\mathcal{F}, \mathcal{G} \in \{\underline{\mathbb{Q}_p}, \mathbb{G}_a\}$, $i = 0, 1, 2$.

Pf: one-by-one compare with Ext^i computed by Jiedong.

e.g. $\mathrm{Ext}^2 = 0$. ,

e.g. $\mathrm{Ext}^1(i_{\infty,*}C, \mathcal{O}_X) = H^0(X, \mathcal{E}xt^1(i_{\infty,*}C, \mathcal{O}_X)) = C.$

$\mathcal{H}om(i_{\infty,*}C, \mathcal{O}_X) = 0$, $\mathcal{E}xt^1(i_{\infty,*}C, \mathcal{O}_X) \cong i_{\infty,*}C.$

$$\mathrm{Hom}_X(\mathcal{O}_X, i_{\infty,*}C) = \mathrm{Hom}_{B\text{-pairs}}((B_e, B_{dR}^+, id), (0, C, 0))$$

$$= \mathrm{Hom}(C, C) = C. \qquad \square$$

$\Rightarrow$ Lemma. if $\mathcal{F}$ & $\mathcal{G}$ fits

$$0 \to V \otimes \mathcal{O}_X \to \mathcal{F}(\mathcal{G}) \to i_{\infty,*}W \to 0$$

then $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^i(T(\mathcal{F}), T(\mathcal{G}))$ $i = 0, 1, 2$.

by five lemma.

for general $\mathcal{F}, \mathcal{G}$ $i = 0, 1$. $(*)$ isom follows from

Lemma: $\forall \mathcal{F} \in \mathrm{Coh}(X)^-$ $\exists$ exact seq in $\mathrm{Coh}(X)^-$

$$0 \to V \otimes \mathcal{O}_X \to \mathcal{F}' \to \mathcal{F} \to 0$$

with $\mathcal{F}' \in \mathrm{Coh}(X)$, fits into

$$0 \to V \otimes \mathcal{O}_X \to \mathcal{F}' \to i_{\infty,*}W \to 0$$

exact in $\mathrm{Coh}(X)$.

Pf: Classification thm. $\mathcal{F} \in \mathrm{Coh}(X)$.

$$\mathcal{F} \cong (\oplus \mathcal{O}(\lambda)) \oplus i_{\infty,*} B_{dR}^+(C_\pi)/t^k.$$

omit details. Lemma 3.7.

e.g. $0 \to \mathcal{O}_X(-k) \xrightarrow{\times t^k} \mathcal{O}_X \to i_{\infty,*} B_{dR}^+/t^k \to 0.$

$\rightsquigarrow$ $0 \to \mathcal{O}_X \to i_{\infty,*} B_{dR}^+/t^k \to \mathcal{O}_X(-k)[1] \to 0.$ $\square$

Remains to show T is exact.

$S \in \mathrm{Perf}_{C^\flat}$ $\rightsquigarrow$ X_S is functorial.

$S \to S'$ proét (covering) $\rightsquigarrow$ $X_S \to X_{S'}$ is also

$\rightsquigarrow \quad \tau : \quad \tilde{X}_{\text{proét}} \longrightarrow (\mathrm{Perf}_{C^b,\text{proét}})^{\sim} \simeq (\mathrm{Perf}_{C,\text{proét}})^{\sim}.$

$D \ni \mathcal{F} \longmapsto R\tau_* \underline{\mathcal{F}}$ complex of sheaves.

complex.

Observation: 0. $\mathcal{F} \in \mathrm{Coh}(X)$, $R^i\tau_*\mathcal{F}$ is the sheafification $S \mapsto H^i(X_S, \mathcal{F}_S)$

1. T is just $R^0\tau_* |_{\mathrm{Coh}(X)^-}$ (use hypercoh spectral seq.).

2. T exact $\Leftrightarrow$ $R^1\tau_* = 0$ on $\mathrm{Coh}(X)^-$.

Concretely. $2 \Leftrightarrow$ $\forall \mathcal{F} \in \mathrm{Coh}(X)$.

1) if slopes of $\mathcal{F} \geq 0$, $R^1\tau_* \underline{\mathcal{F}} = 0$

2) if slopes of $\mathcal{F} < 0$, $R^0\tau_* \underline{\mathcal{F}} = 0$. sheaf.

$\underline{\text{Pf}}$ classification thm.

can assume $\mathcal{F} = \mathcal{O}(\lambda)$

$\mathcal{F} = i_{x,\infty} B^+_{dR}(C_x)/t^k$. ✓. $H^1(X_S, \mathcal{F}_S) = 0$

recall $Y_S = \bigcup_{m,n \geq 1} Y_{S,m,n}$

1) each $Y_{S,m,n}$ affinoid sous perfectoid (Kedlaya–Hansen).

$\Rightarrow H^i(Y_{S,m,n}, \mathcal{E}) = 0$ $\quad \mathcal{E}$ v.b.

2). transition maps. $\mathcal{O}(Y_{S,m,n})$ has dense image.

last week. $\Rightarrow R^1\lim = 0$ & $H^i(Y_S, \mathcal{E}) = 0$. $i>0$.

$H^i(X_S, \mathcal{E})$ is computed by

$$[\mathcal{O}(\mathcal{E}) \xrightarrow{\varphi - id} \mathcal{O}(\mathcal{E})]$$

$\underline{H^i(X_S, \mathcal{E}) \text{ are } \mathbb{Q}_p\text{-v.s}}$

- $H^0(\mathcal{O}(\lambda)) = B(S)^{\varphi^h = p^d}$ $\quad \lambda = \frac{d}{h}$

when $\lambda < 0$.

discussed in Koji lecture.

Newton polygon method $\Rightarrow$ $B(S)^{\varphi^h = p^d} \overset{\downarrow}{=} W(R^+)^{\varphi^h = p^d} \overset{d<0}{=} 0$.

- $H^1(\mathcal{O}(\lambda)) = 0$. $\lambda > 0$.

replace X_S by $X_{\mathbb{Q}_{p^h}, S}$. reduce to show $\lambda = d \in \mathbb{Z}$.

$$B \xrightarrow{\mathrm{id} - p^{-d}\varphi} B.$$

surjective. KL. Prop 6.2.2.

- $H^1(X_S, \mathcal{O}_{X_S}) = 0$ $\overbrace{, \text{ apply}}^{\text{if } S = \mathrm{Spa}(R,R^+) \text{ is sympathetic}}$ $\Gamma(X_S, -)$ to.

$$0 \to \mathcal{O}_{X_S} \xrightarrow{\times t} \mathcal{O}_{X_S}(1) \longrightarrow i_{\infty *}\mathcal{O}_{S^\#} \to 0.$$

exactness from. $S = \mathrm{Spa}(R, R^+)$ sympathetic.

$$t \in H^0(X_S, \mathcal{O}_{X_S}(1)) = B(S)^{\varphi=1}.$$

and FES for sympathic. rings.

$\square$