Lect 13 Overview and local syntomic computations 12/12/22

Colmez - Niziol Syntomic complexes & p-adic nearby cycles, Invent.

§1. Introduction

setup k perfect field of char $p>0$, $F = W(k)[1/p]$

K/F fin tot. ramif ext of deg e $\varpi \in \mathcal{O}_K$ unif, $E(X_0) \in \mathcal{O}_F[X_0]$ min poly.

Thm (C_{st}) Let $\mathfrak{X}$ be a proper semistable (p-adic formal) scheme $/\mathcal{O}_K$

$\exists$ nat isom
$$\begin{array}{ccc} H^j_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} & \simeq & H^j_{HK}(\mathfrak{X}) \otimes_{K_0} B_{st} \\ \downarrow & & \downarrow \\ H^j_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} & \simeq & H^j_{dR}(\mathfrak{X}_K) \otimes_K B_{dR} \end{array}$$

Hyodo-Kato $H_{\log cris}[1/p]$

comp w/ add structures

Rem. many proofs (at least $\mathfrak{X}$ scheme)! Tsuji syntomic; Faltings alm étale; Niziol K-theory } + Poincaré duality

Colmez - Niziol syntomic + Banach - Colmez spaces

naive strategy define a cohom $H^j_{syn}(r) := H^j_{syn}(\mathfrak{X}_{\mathcal{O}_{\bar{K}}}, r)_{\mathbb{Q}}$ that sits in
$$\cdots \to \frac{H^{j-1}_{dR} \otimes_K B^+_{dR}}{Fil^r} \longrightarrow H^j_{syn}(r) \longrightarrow \left(H^j_{HK} \otimes_F B^+_{st}\right)^{\varphi=p^r, N=0} \longrightarrow \frac{H^j_{dR} \otimes_K B^+_{dR}}{Fil^r} \to \cdots$$

Show: ① $\exists\ \alpha_r : H^j_{syn}(r) \xrightarrow{\sim} H^j_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p(r))$ if $0 \le j \le r$

② $$0 \to H^j_{syn}(r) \to (H^j_{HK} \otimes B^+_{st})^{\varphi=p^r, N=0} \longrightarrow \frac{H^j_{dR} \otimes B^+_{dR}}{Fil^r} \to 0$$

exact if $0 \le j < r$.

王老师 ⟸ BC spaces + Dim theory + $H^j_{syn}(r)$ f.d. $\mathbb{Q}_p$-vs ⟸ ① + Scholze's finiteness

today + next week: local computation necessary for ①

final week: global ①+②

⟸ Mao 老师

§2. local syntomic computations

p-adic compl poly

Today $R := \mathcal{O}_K\{X_1,\dots,X_d,\frac{\varpi}{X_1\cdots X_d}\} = \mathcal{O}_K\{X_1,\dots,X_{d+1}\}/(X_1\cdots X_{d+1}-\varpi)$

note. Colmez-Nizioł $R=$ p-adic compl étale ext of $\mathcal{O}_K\{X_1,\dots,X_d,\frac{\varpi^{h}}{X_1\cdots X_d},Y_1^{\pm1},\dots,Y_a^{\pm1},Z_1,\dots,Z_b\}$ $\quad 1\le h\le e$

Def. $r_\varpi^+ := S := \mathcal{O}_F[[X_0]] \xrightarrow{X_0\mapsto\varpi} \mathcal{O}_K$, $\ker = (E(X_0))$

$R_\varpi^+ := (p,X_0)$-compl of $\mathcal{O}_F[X_0,X_1,\dots,X_d,\frac{X_0}{X_1\cdots X_d}]$

$$\begin{tikzcd} \mathrm{Spf}\,R \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spf}\,R_\varpi^+ \arrow[d] \\ \mathrm{Spf}\,\mathcal{O}_K \arrow[r, hook] & \mathrm{Spf}\,r_\varpi^+ \arrow[d] \\ & \mathrm{Spf}\,\mathcal{O}_F \end{tikzcd}$$

Want: "PD-nbd" (PD-envelope)

exer $E(X_0)\equiv X_0^e \mod p$

$r_\varpi^{PD} :=$ p-adic compl of $r_\varpi^+\left[\frac{E(X_0)^j}{j!} : j\ge 0\right] \overset{\text{exer}}{=} r_\varpi^+\left[\frac{X_0^\ell}{[\ell/e]!} : \ell\ge 0\right]^{\wedge(p)}$

$R_\varpi^{PD} := r_\varpi^{PD}\,\hat{\otimes}_{r_\varpi^+}\,R_\varpi^+ = \left(\text{p-adic PD-envelope of } R_\varpi^+ \twoheadrightarrow R\right)$

p-adic compl

Def. (additional strs on R_ϖ^{PD})

- filtration $F^rR_\varpi^{PD}$:

$$\begin{tikzcd} R_\varpi^{PD} \arrow[r, hook] & R_\varpi^{PD}[1/p]^{\wedge(E(X_0))} \arrow[r, "\text{exer}", "\simeq"'] & R[1/p][[X_0-\varpi]] \\ F^rR_\varpi^{PD} \arrow[u, phantom, "\cup"] \arrow[rr, leftrightarrow] \arrow[urr, phantom, "\square"] & & (X_0-\varpi)^r\cdot R[1/p][[X_0-\varpi]] \arrow[u, phantom, "\cup"] \end{tikzcd}$$

- Kummer Frobenius $\varphi=\varphi_{Kum}$:

  $\exists!$ cont ring endom $\varphi_{Kum}: R_\varpi^{PD}\longrightarrow R_\varpi^{PD}$ s.t. $\varphi|_{\mathcal{O}_F}=W(\mathrm{Frob}_p)$, $\varphi(X_j)=X_j^p$ $\quad 0\le j\le d$

- log diff: $\Omega^1_{R_\varpi^{PD}} = \bigoplus_{j=0}^{d} R_\varpi^{PD}\frac{dX_j}{X_j}$, $\quad \Omega^n_{R_\varpi^{PD}} = \bigwedge^n\Omega^1_{R_\varpi^{PD}} = \bigoplus_{j_1<\dots<j_n} R_\varpi^{PD}\frac{dX_{j_1}}{X_{j_1}}\wedge\dots\wedge\frac{dX_{j_n}}{X_{j_n}}$

  $\cup$

  $F^r\Omega^n := \bigoplus F^rR_\varpi^{PD}\left(\text{—〃—}\right)$

  $\varphi=\varphi_{Kum} \curvearrowright \Omega^n_{R_\varpi^{PD}}$ coeff wise!

$\to (\Omega^\bullet_{R_\varpi^{PD}}, d)$ log de Rham cpx

obs. · $F^r\Omega^\bullet := \left(F^rR_\varpi^{PD}\xrightarrow{d} F^{r-1}\Omega^1_{R_\varpi^{PD}} \xrightarrow{d} F^{r-2}\Omega^1_{R_\varpi^{PD}} \to \cdots\right)$ subcpx

· $p^\bullet\varphi_{Kum}: \Omega^\bullet \to \Omega^\bullet$ endom of cpx $\leftarrow$ $d\,\varphi(X_j) = d\,X_j^p = p\,X_j^p\frac{dX_j}{X_j} = p\,\varphi_{Kum}\left(X_j\frac{dX_j}{X_j}\right)$ $\quad dX_j$

Def. $\mathrm{Syn}(R,r) := \mathrm{Kum}(R_\varpi^{PD},r) := \mathrm{Cone}\left(F^r\Omega^\bullet_{R_\varpi^{PD}} \xrightarrow{p^r-p^\bullet\varphi_{Kum}} \Omega^\bullet_{R_\varpi^{PD}}\right)[-1]$

$\mathrm{Syn}^\bullet = \begin{matrix} F^r\Omega^\bullet \\ \oplus \\ \Omega^{\bullet-1}\end{matrix} \circlearrowleft d = \begin{pmatrix} d & 0 \\ p^r-p^\bullet\varphi_{Kum} & d \end{pmatrix}$ $\quad$ local syntomic cpx (w.r.t. fixed chart)

$$H^*_{syn}(R,r) := H^*(\mathrm{Syn}(R,r))$$

Goal of today + next week. Relate $\tau_{\le r}\mathrm{Syn}(R,r)$ & $\tau_{\le r} R\Gamma_{cont}(Gal_{R[1/p]}, \mathbb{Z}_p(r))$

Switch: §3 PD-nbd ⇝ annulus, §4 Kummer ⇝ cyclotomic → today

"$\tau_{\le r}$ cyclotomic cpx" ⟷ ⇝ next week "(φ,Γ)-modules"

§3. switch from PD-nbd to annulus

Recall $r^+_\varpi = \mathcal{O}_F[[x_0]]$ open unit disc $|x_0|<1 \iff v_p(x_0)>0$.

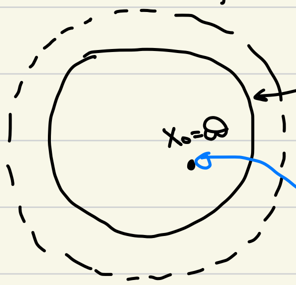
Def. For $0<u\le v$,

$$r^{[u]}_\varpi := r^+_\varpi\left[\frac{x_0^j}{p^{\lfloor uj/e\rfloor}} : j\ge 0\right]^{\wedge(p)} \quad \text{" } v_p(x_0)\ge \tfrac{1}{e}u \text{ "}$$

$$r^{[u,v]}_\varpi := r^+_\varpi\left[\frac{x_0^j}{p^{\lfloor uj/e\rfloor}}, \frac{p^{\lfloor vj/e\rfloor}}{x_0^j} : j\ge 0\right]^{\wedge(p)} \quad \text{" } \tfrac{1}{e}v\ge v_p(x_0)\ge \tfrac{1}{e}u \text{ "}$$

Obs.

r^+_ϖ



$r^{PD}_\varpi \sim r^{[1/p-1]}_\varpi$ ← $\frac{1}{e}\lim_{j\to\infty}\frac{v_p(j!)}{j} = \frac{1}{e}\frac{1}{p-1}$

$v_p(x_0)\ge \frac{1}{e}\cdot\frac{1}{p-1}$

$v_p(x_0)=\frac{1}{e}$

Set $R^{[u]}_\varpi := r^{[u]}_\varpi \hat{\otimes}_{r^+_\varpi} R^+_\varpi$, $R^{[u,v]}_\varpi := r^{[u,v]}_\varpi\hat{\otimes}_{r^+_\varpi} R^+_\varpi$

→ · filtr: when $u\le 1\le v$, can define $F^rR^{[u]}_\varpi$, $F^rR^{[u,v]}_\varpi$ as before

$\varphi_{Kum}: R^{[u]}_\varpi \longrightarrow R^{[u/p]}_\varpi \xrightarrow{res} R^{[u]}_\varpi$

$u\le v/p$ $\quad R^{[u,v]}_\varpi \longrightarrow R^{[u/p,v/p]}_\varpi \xrightarrow{res} R^{[u,v/p]}_\varpi$

Def. $\mathrm{Kum}(R^{[u]}_\varpi, r) := \mathrm{Cone}\left(F^r\Omega^\bullet_{R^{[u]}_\varpi} \xrightarrow{p^r-p^\bullet\varphi_K} \Omega^\bullet_{R^{[u]}_\varpi}\right)[-1]$

$\mathrm{Kum}(R^{[u,v]}_\varpi, r) := \mathrm{Cone}\left(F^r\Omega^\bullet_{R^{[u,v]}_\varpi} \xrightarrow{p^r-p^\bullet\varphi_K} \Omega^\bullet_{R^{[u,v/p]}_\varpi}\right)[-1]$

Prop. 1 (1) if $\frac{1}{p-1}\le u\le 1$, then $R^{PD}_\varpi \hookrightarrow R^{[u]}_\varpi$ induces

$$\tau_{\le r}\mathrm{Kum}(R^{PD}_\varpi, r) \longrightarrow \tau_{\le r}\mathrm{Kum}(R^{[u]}_\varpi, r) \quad p^{4r}\text{-qisom.}$$

$p^{4r}\mathcal{H}^i(\ker) = p^{4r}\mathcal{H}^i(\mathrm{Cok})=0$

(2) $\exists$ nat p^{2r}-qisom $\tau_{\le r}\mathrm{Kum}(R^{[u]}_\varpi, r)$ & $\tau_{\le r}\mathrm{Kum}(R^{[u,v]}_\varpi, r)$

sketch (1) SES of cpxes

$$\begin{array}{ccccccccc}
0 & \to & \Omega^{\bullet-1}_{PD} & \to & \mathrm{Kum}(PD,r) & \to & F^r\Omega^\bullet_{PD} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^{\bullet-1}_{[u]} & \to & \mathrm{Kum}([u],r) & \to & F^r\Omega^\bullet_{[u]} & \to & 0
\end{array}$$

writing $p^r - p^\bullet\varphi = p^\bullet(p^{\overbrace{r-\bullet}^{s}} - \varphi)$ reduces (1) to: (omit)

<u>Lem.</u> if $s \ge 0$, $\frac{1}{p-1} \le u \le 1$, then

$p^s - \varphi$ induces $F^r R^{[u]} / F^r R^{PD} \longrightarrow R^{[u]} / R^{PD}$ $\quad p^{s+r}$-isom

☹ Standard: p^r-inj ($\Rightarrow p^{s+r}$-inj): By def $F^r R^{PD} = R^{PD} \cap F^r R^{[u]}$

STP: $x \in R^{[u]}$, $(p^s - \varphi)(x) \in R^{PD} \Rightarrow p^s x \in R^{PD}$

<u>obs.</u> $u \le 1 \Rightarrow u/p \le 1/p < 1/(p-1) \Rightarrow \varphi(R^{[u]}) \subset R^{PD}$

so $p^s x = \underbrace{(p^s-\varphi)(x)}_{\in R^{PD}} + \underbrace{\varphi(x)}_{\in R^{PD}\ \text{obs}} \in R^{PD}$ // $\Rightarrow$ (1)

p^{s+r}-surj (omit)

<u>main tool for (2)</u> <u>Fact.</u> $\exists$ $\mathbb{Z}_p$-lin hom $\psi = \psi_{Kum} : R^{[u]}_{\mathfrak{S}} \to R^{[pu]}_{\mathfrak{S}}$ s.t. $\psi \circ \varphi = \mathrm{id}$.

$\psi : R^{[u/p, v]} \longrightarrow R^{[u, pv]}$ $\quad$ ($\Rightarrow \psi$ surj)

Look at

$$\begin{tikzcd}
F^r\Omega^\bullet_{[u]} \arrow[r, "p^r - p^\bullet\varphi"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^\bullet_{[u]} \arrow[d, "\psi"] & & \mathrm{Kum}([u], r) \arrow[d, "(A)"] \\
F^r\Omega^\bullet_{[u]} \arrow[r, "p^r\psi - p^\bullet"] \arrow[d, "\mathrm{res}"] & \Omega^\bullet_{[pu]} \arrow[d, "\mathrm{res}"] & \mathrm{Cone}[-1] \arrow[r, squiggly] & \mathrm{Kum}^\psi([u], r) \arrow[d, "\mathrm{res}\ (B)"] \\
F^r\Omega^\bullet_{[u,v]} \arrow[r, "p^r\psi - p^\bullet"] & \Omega^\bullet_{[pu,v]} & & \mathrm{Kum}^\psi([u,v], r) \\
F^r\Omega^\bullet_{[u,v]} \arrow[u, equal] \arrow[r, "p^r - p^\bullet\varphi"] & \Omega^\bullet_{[u,v/p]} \arrow[u, "\psi"] & & \mathrm{Kum}([u,v], r) \arrow[u, "(C)"]
\end{tikzcd}$$

Can show (A), (C) qisom by checking acyclicity of $\Omega^{\bullet, \psi=0}$

$\tau_{\le r}(B)$ p^{2r}-qisom $\leftarrow$ $p^r\psi - p^\bullet = -p^\bullet(1 - \underbrace{p^{r-\bullet}\psi}_{})$

ψ top nilp $\Rightarrow$ invertible if $r - \bullet \ge 0$

§4. Switch from Kummer to cyclotomic

<u>need add assump on K</u> Fix $(\zeta_{p^n})_n$ in $\bar{K}$. $F_n = F(\zeta_{p^n})$.

$$e\left(\begin{array}{c} K \\ | \\ F_i \\ | \\ F \end{array}\right)\begin{array}{l} f \\ \\ (p-1)p^{i-1} \end{array}$$

$i := \max\{n : \zeta_{p^n} \in K\}$ $\quad K_n := K(\zeta_{p^{n+i}})$

$\delta_K := e\, v_p(\mathfrak{D}_{K/F_i})$

Say K <u>contains enough roots of unity</u> if $\delta_K < \frac{e}{2p} - f$ $\quad\leftarrow$ K_∞/F_∞ alm étale; true for K_n, $n \gg 0$ in place of K

We assume this in what follows:

$\zeta := \zeta_{p^i}$, $\zeta - 1 \in \mathcal{O}_F$ unif

$$\mathcal{O}_F[[T,X_0]]/(Q(X_0,T)) \overset{\exists}{\simeq} r^+_\varpi = \mathcal{O}_F[[X_0]] \xrightarrow{X_0 \mapsto \varpi} \mathcal{O}_K$$

$$\cup$$

$$r^+_{\zeta-1} = \mathcal{O}_F[[T]] \xrightarrow[T \mapsto \zeta-1]{} \mathcal{O}_{F_i}$$

$$\cup$$

$$\mathcal{O}_F$$

$$r^+_\varpi = \bigoplus_{\ell=0,\dots,f-1} r^+_{\zeta-1} T^\ell$$

Choose $Q(X_0,T) = X_0^f + A_{f-1}(T) X_0^{f-1} + \cdots + A_0(T)$ in $\mathcal{O}_F[X_0,T]$

s.t. $\begin{cases} A_j(T) \in T\,\mathcal{O}_F[T], \ A_0 \in \mathcal{O}_F^\times \cdot T + T^2 \mathcal{O}_F[T] \\ Q(X_0, \zeta-1) \text{ is min poly of } \varpi \ /F_i \end{cases}$

Set $R_{\zeta-1,\square} := \mathcal{O}_F[[T]]\{X_1,\dots,X_d\} \to \dfrac{R^+_{\zeta-1,\square}\left\{\frac{X_0}{X_1\cdots X_d}\right\}}{(Q(X_0,T))} \simeq R^+_\varpi$

Def/Prop. cyclotomic Frob $\varphi_{cycl} \circlearrowright R^+_{\zeta-1,\square}$ by $\begin{cases} T \mapsto (1+T)^p - 1 \\ X_j \mapsto X_j^p \quad 1 \le j \le d \end{cases}$

→ $\exists$ nat extend $\varphi_{cycl} : R^+_\varpi \to R^+_\varpi$

$$R_\varpi^{[u,v]} \to R_\varpi^{[u/p,v/p]} \xrightarrow{res} R_\varpi^{[u,v/p]}$$

if $pu \le v < \dfrac{e}{2f([d_K/f]+1)}$

Def. $\mathrm{Cycl}(R_\varpi^{[u,v]}, r) := \mathrm{Cone}\left(F^r\Omega^\bullet_{R_\varpi^{[u,v]}} \xrightarrow{p^r - p^\bullet \varphi_{cycl}} \Omega^\bullet_{R_\varpi^{[u,v/p]}}\right)[-1]$

Prop. 2 $(p \ge 3)$ $\exists$ nat qisom b/w $\mathrm{Kum}(R_\varpi^{[u,v]}, r)$ & $\mathrm{Cycl}(R_\varpi^{[u,v]}, r)$

$2^{(d+1)}$

idea. Set $R_1 = R_2 = R_\varpi^{[u,v]}$

R_3 = p-adic log PD-envelope of $R_1 \otimes_{\mathbb{Z}_p} R_2 \xrightarrow{mult_T} R_\varpi^{[u,v]}$

$= (R_1 \otimes R_2)\left[V_0,\dots,V_d, \dfrac{(V_j-1)^k}{k!} \ k \ge 0\right]^{\wedge(p)}$ $\qquad V_j = \dfrac{X_j \otimes 1}{1 \otimes X_j}$

→ $F^r R_3$, $\varphi_{R_3} = \varphi_{Kum} \otimes \varphi_{cycl} : R_3 \to R_3^{[u,v/p]}$

Claim filtered PD Poincaré lemma: $\qquad \Omega^1_{R_3} = p_1^*\Omega^1_{R_1} \oplus p_2^*\Omega^1_{R_2}$

$$F^r\Omega^\bullet_{R_1} \to F^r\Omega^\bullet_{R_3} := (F^rR_3 \to F^{r-1}\Omega^1_{R_3} \to F^{r-2}\Omega^2_{R_3} \to \cdots)$$

qisom $\quad$ qis $\leftarrow F^r\Omega^\bullet_{R_2}$

→ $\mathrm{Kum}([u,v],r) \xrightarrow[qis]{} \mathrm{Cone}\left(F^r\Omega^\bullet_{R_3} \xrightarrow{p^r - p^\bullet\varphi_{R_3}} \Omega^\bullet_{R_3^{[u,v/p]}}\right)[-1] \xleftarrow[qis]{} \mathrm{Cycl}([u,v],r)$ //

$$R_3 = R_1\langle V_0 - 1, \dots, V_d - 1\rangle^{PD}$$