**Setup**: $\mathcal{O}_K$ C.D.V.R. $\quad K = \mathrm{Frac}(\mathcal{O}_K)$. $\quad k$ residue field of char. $p$

$\mathcal{O}_F = W(k) \quad F = \mathrm{Frac}(\mathcal{O}_F)$. $\quad \varphi$ ab. Frob. on $W(k)$

$\varpi$ uniformizer of $\mathcal{O}_K$.

$$R_\square := \frac{\mathcal{O}_F\{X_1^{\pm1}, \dots, X_a^{\pm1}, X_{a+1}, \dots, X_{d+1}\}}{(X_{d+1}X_{a+1}\cdots X_{a+b} - \varpi)} \qquad a+b \le d$$

$\downarrow$

$R$ $\quad$ $p$-adic completion of an étale $R_\square$-alg.

choose crystalline coordinate for $R$.

$$R^+_\varpi := \mathcal{O}_F[[X_0]] \twoheadrightarrow \mathcal{O}_K, \quad X_0 \longmapsto \varpi.$$

$$R^+_{\varpi,\square} = \overbrace{\mathcal{O}_F[X_0, X_1^{\pm1}, \dots, X_a^{\pm1}, X_{a+1}, \dots, X_{d+1}, \tfrac{X_0}{X_{d+1}X_{a+1}\cdots X_{a+b}}]}^{(p, X_0)\text{-adic}}$$

$$\begin{array}{cc} X_0 & R^+_{\varpi,\square} \\ \downarrow & \downarrow \\ \varpi & R_\square \end{array}$$

$\ker = (P_\varpi(X_0))$.

unique étale lift of $R$ over $R^+_{\varpi,\square}$:

$$\begin{array}{ccc} R^+_\varpi & \longleftarrow & R^+_{\varpi,\square} \\ \downarrow & & \downarrow \\ R & \longleftarrow & R_\square \end{array}$$

complete for $(p, P_\varpi(X_0))$-adic top.

$R^{PD}_\varpi$ $\quad$ p.d. envelope of $R$ in $R^+_\varpi$ $\qquad e = [K:F]$

$\circlearrowright$

$\varphi_{Kum}: X_i \longmapsto X_i^p \quad 0 \le i \le d+1.$

$$\mathrm{Syn}(R, r) := \mathrm{Cone}\Big( F^r\Omega^\bullet_{R^{PD}_\varpi} \xrightarrow{p^r - p^\bullet\varphi_{Kum}} \Omega^\bullet_{R^{PD}_\varpi} \Big)[-1]$$

where $\Omega^\bullet_{R^{PD}_\varpi} := R^{PD}_\varpi \otimes_{R^+_\varpi} \Omega^\bullet_{R^+_\varpi/\mathcal{O}_F}$.

(*). Switch from P.D envelope to annulus.

$0 < u \le v \qquad R^{[u]}_\varpi := \overbrace{R^+_\varpi\Big[\frac{X_0^j}{p^{[\frac{uj}{e}]}} : j \ge 0\Big]}^{(p)\text{-adic completion}} \qquad v_p(X_0) \ge \frac{1}{e}u.$

$p$-adic.

$$r_\varpi^{[u,v]} := r_\varpi^+ \left[ \frac{X_0^j}{p^{[uj/e]}} , \frac{p^{[vj/e]}}{X_0^j} : j \geq 0 \right]$$

$$\text{“} \tfrac{1}{e} v \geq v_p(X_0) \geq \tfrac{1}{e} u \text{”}$$

$$R_\varpi^{[u]} := r_\varpi^{[u]} \hat{\otimes}_{r_\varpi^+} R_\varpi^+ \qquad R_\varpi^{[u,v]} := r_\varpi^{[u,v]} \hat{\otimes}_{r_\varpi^+} R_\varpi^+$$

Filtration : by order of vanishing at $X_0 = \varpi$.

$$\varphi_{Kum} : R_\varpi^{[u]} \longrightarrow R_\varpi^{[u/p]} \xrightarrow{res} R_\varpi^{[u]} \qquad \text{if } u < \frac{v}{p}$$

$$R_\varpi^{[u,v]} \longrightarrow R_\varpi^{[u/p, v/p]} \xrightarrow{res} R_\varpi^{[u, v/p]}.$$

**Def** : $Kum(R_\varpi^{[u]}, r) := Cone\left( F^r \Omega^\bullet_{R_\varpi^{[u]}} \xrightarrow{p^r - p^\bullet \varphi_{Kum}} \Omega^\bullet_{R_\varpi^{[u]}} \right)[-1].$

$$Kum(R_\varpi^{[u,v]}, r) := Cone\left( F^r \Omega^\bullet_{R_\varpi^{[u,v]}} \xrightarrow{\;*\;} \Omega^\bullet_{R_\varpi^{[u,v]}} \right)[-1]$$

The inclusion $R_\varpi^{PD} \to R_\varpi^{[u]}$ induces a $p^{6r}$-qis for $\frac{1}{p-1} \leq u \leq 1$.

$$\begin{array}{ccc} \tau_{\leq r} Kum(R_\varpi^{PD}, r) & \longrightarrow & \tau_{\leq r} Kum(R_\varpi^{[u]}, r). \\ & & \big\downarrow {\scriptstyle p^{2r}\text{-qis}} \\ & & \tau_{\leq r} Kum(R_\varpi^{[u,v]}, r). \end{array}$$

Use $\psi_{Kum}$

(2) Switch from Kummer to cyclotomic.

Fix a system of $(\zeta_{p^n})$ in $\bar{K}$, $F_n = F(\zeta_{p^n})$

$i := \max\{n : \zeta_{p^n} \in K\}$. $K_n := K(\zeta_{p^{n+i}})$

$\zeta := \zeta_{p^i}$ $\zeta - 1 \in O_{F_i}$ uniformizer $\delta_K := e \cdot v_p(\delta_{K/F_i})$.

If $\delta_K < \frac{e}{2p} - f$, then say $K$ contains enough roots of unity

$f = [K:F_i]$

Set $R^+_{\xi-1,\square} := O_F[[T]]\{X_1, \dots, X_d\}$

$$\begin{array}{ccc} O_F[[T,X_0]]/(Q(X_0,T)) \cong r^+_{\varpi} & \twoheadrightarrow & O_K \\ \uparrow & & \cup \\ O_F[[T]] =: r^+_{\xi-1} & \twoheadrightarrow & O_{F_i} \\ T & \mapsto & \xi - 1 \end{array}$$

$$\circlearrowleft \quad \varphi_{cycl}: \quad T \mapsto (1+T)^p - 1, \qquad X_j \mapsto X_j^p \quad 1 \le j \le d$$

$$R^+_{\varpi,\square} = \frac{R^+_{\xi-1,\square}\{X_0, X_{d+1}, X_{d+2}\}}{\left(Q(X_0,T),\ X_{d+1}X_{a+1}\cdots X_{a+b} - X_0^h,\ X_{d+2}X_1\cdots X_a - 1\right)}$$

$h \in [1,e]$

$$\varphi_{cycl}: R^+_{\varpi,\square} \to R^+_{\varpi,\square}.$$

$$\varphi_{cycl}: R_\varpi^{[u,v]} \to R_\varpi^{[u/p,\, v/p]} \xrightarrow{res} R_\varpi^{[u,\, v/p]}.$$

**Def**: $\mathrm{Cycl}(R_\varpi^{[u,v]}, r) := \mathrm{Cone}\left( F^r \Omega^\bullet_{R_\varpi^{[u,v]}} \xrightarrow{p^r - p^\bullet \varphi_{cycl}} \Omega^\bullet_{R_\varpi^{[u,v/p]}} \right)[-1]$

**Prop**: $\mathrm{Kum}(R_\varpi^{[u,v]}, r)$ & ~~Ko~~ $\mathrm{Cycl}(R_\varpi^{[u,v]}, r)$ are $2^{2(d+1)}$-qiso.

comes from filtered Poincaré lemma.

(3) Transform $\mathrm{Cycl}(R_\varpi^{[u,v]}, r)$ into "Koszul complex"

basis of $F^{r-i}\Omega^i_{R_\varpi^{[u,v]}}$: $\omega_0 = \frac{dT}{1+T}$, $\omega_j = \frac{dX_j}{X_j}$ for $1 \le j \le d$

$\forall\ \underline{j} = (j_1, \dots, j_i) \in J_i = \{0 \le j_1 < \cdots < j_i \le d\}$

Set $\omega_{\underline{j}} = \omega_{j_1} \wedge \cdots \wedge \omega_{j_i}$.

Then $F^{r-i}\Omega^i_{R_\varpi^{[u,v]}} = \left\{ \sum_{\underline{j} \in J_i} x_{\underline{j}} \cdot \omega_{\underline{j}} : x_{\underline{j}} \in F^{r-i} R_\varpi^{[u,v]} \right\}$

$$
\begin{tikzcd}
F^{r-i}\Omega^{i}_{R_\varpi^{[u,v]}} \arrow[r, phantom, "\cong"] \arrow[d, "d"] & \left(F^{r-i}R_\varpi^{[u,v]}\right)^{J_i} \arrow[d, "(\partial_j)"] \\
F^{r-i-1}\Omega^{i+1}_{R_\varpi^{[u,v]}} \arrow[r, phantom, "\cong"] & \left(F^{r-i-1}R_\varpi^{[u,v]}\right)^{J_{i+1}}
\end{tikzcd}
$$

$$\mathrm{Kos}(\partial, F^rR_\varpi^{[u,v]}) := F^rR_\varpi^{[u,v]} \xrightarrow{(\partial_j)} \left(F^{r-1}R_\varpi^{[u,v]}\right)^{J_1} \to \cdots$$

$$\mathrm{Cyc}(R_\varpi^{[u,v]}, r) \cong \mathrm{Kos}(\varphi, \partial, F^rR_\varpi^{[u,v]})$$

$$:= \left[\underline{\mathrm{Kos}(\partial, F^rR_\varpi^{[u,v]})} \xrightarrow{p^r - p^i\varphi_{[u,v]}} \underline{\mathrm{Kos}(\partial, R_\varpi^{[u,v/p]})}\right]$$

homotopy limit

Separate the geometric variable & arith. variable

$$
= \left[
\begin{tikzcd}
\mathrm{Kos}(\partial', F^rR_\varpi^{[u,v]}) \arrow[r, "p^r - p^i\varphi_{[u,v]}"] \arrow[d, "\partial_0"] & \mathrm{Kos}(\partial', R_\varpi^{[u,v/p]}) \arrow[d, "\partial_0"] \\
\mathrm{Kos}(\partial', F^{r-1}R_\varpi^{[u,v]}) \arrow[r] & \mathrm{Kos}(\partial', R_\varpi^{[u,v/p]})
\end{tikzcd}
\right].
$$

(4). embedding into ~~period~~ ring of periods

$\overline{R}$ max. ext. of $R$ $\underline{\text{unramified}}$ outside $X_{a+b+1} \cdots X_d = 0$

$\underline{\text{after inverting } p}$.

$G_R := \mathrm{Gal}(\overline{R}/R)$. $\quad v_p$ the spectral valuation on $\overline{R}[\frac{1}{p}]$.

$S = K$ or $R[\frac{1}{p}]$ : $\quad C(\overline{S})$ the completion of $\overline{S}$ for $v_p$. perfectoid alg.

$$C(\overline{S}) \supset C^+(\overline{S}) = \{x \in C(\overline{S}) : v_p(x) \geq 0\}$$

$E_{\overline{S}} = C(\overline{S})^\flat \qquad A_{\overline{S}} = W(E_{\overline{S}})$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\tilde{E}_{\bar S}^{+} = \{x \in \tilde{E}_{\bar S} : v_E(x) \geq 0\} \qquad A_{\bar S}^{+} = W(\tilde{E}_{\bar S}^{+})$$

$$= C^{+}(S)^{\flat}$$

p-adic completion

$$A_{cr}(\bar S) = \widehat{A_{\bar S}^{+}\left[\frac{(p-[p^\flat])^k}{k!} : k \in \mathbb{N}\right]}$$

$$= A_{\bar S}^{+}\left[\frac{[p^\flat]^k}{k!} : k \in \mathbb{N}\right].$$

If $v > 0$, $$A_{\bar S}^{[0,v]} := \left\{ \sum_{k\in\mathbb{N}} p^k [x_k] : v \cdot v_E(x_k) + \overset{?}{k} \to +\infty \text{ as } k \to +\infty \right\}$$

$$\cup$$

$$A_{\bar S}^{[0,v]+} := \left\{ x \in A_{\bar S}^{[0,v]} : v \cdot v_E(x_k) + \overset{?}{k} \geq 0 \ \forall k \in \mathbb{N} \right\}$$

If $u > 0$, & $\beta \in \tilde{E}_{R}^{+}$ s.t. $v_E(\beta) = \frac{1}{u}$, then

set $A_{\bar S}^{[u]} = \widehat{A_{\bar S}^{+}\left[\frac{[\beta]}{p}\right]}$ → p-adic

If $0 < u \leq v$, $\alpha, \beta \in \tilde{E}_R^{+}$ s.t. $v_E(\alpha) = \frac{1}{v}$, $v_E(\beta) = \frac{1}{u}$.

$$A_{\bar S}^{[u,v]} := \widehat{A_{\bar S}^{+}\left[\frac{p}{[\alpha]}, \frac{[\beta]}{p}\right]}$$ → p-adic.

are subring of $B_{dR}^{+}(\bar S)$. $\qquad \varphi$ action on $A_{\bar S}^{\boxed{deco}}$

(p, X_0)-adic $\qquad$ Filtration

$$R_{\varpi,\square}^{+} := \widehat{O_F[X_0, X_1^{\pm1}}, \dots X_a^{\pm1}, X_{a+1}, \dots X_{d+1}, \frac{X_0}{X_{d+1}X_{a+1}\cdots X_{a+d}}]$$

$\downarrow$ [illegible] $\qquad X_0 \mapsto T \qquad X_i \mapsto T \qquad 1 \leq i \leq d.$

$A_{\overline{R}}$ $\quad [\varpi^b]$ $\quad [X_i^b]$

$\rightsquigarrow R_{\varpi}^+ \longrightarrow A_{\overline{R}}$ $\quad \left( R_{\varpi}^+ / R_{\varpi,\square}^+ \text{ is étale.} \right)$

$\rightsquigarrow \begin{cases} R_{\varpi}^{PD} \xrightarrow{\varphi_{\mathrm{Kum}}} A_{\mathrm{cr}}(\overline{R}) \quad \varphi \\ R_{\varpi}^{[u]} \longrightarrow A_{\overline{R}}^{[u]} \\ R_{\varpi}^{[u,v]} \longrightarrow A_{\overline{R}}^{[u,v]} \end{cases}$

commutes with $\varphi_{\mathrm{Kum}}$ & $\varphi$

filtration. (if $1 \notin [u,v]$)

$1 \in [u,v]$ $\quad$ no filtration

P29. §2.5

Cyclotomic embedding

$$R_{\square}^{\mathrm{cycl}} := \mathcal{O}_{F_i}\{X_1, \dots, X_d\} \longrightarrow R_{\square,n}^{\mathrm{cycl}} = \mathcal{O}_{F_{i+n}}\{X_i^{p^{-n}} : 1 \le i \le d\}$$

$$R_{\square,\infty}^{\mathrm{cycl}} = \bigcup_n R_{\square,n}^{\mathrm{cycl}}$$

$R_{\square,\infty}^{\mathrm{cycl}}[\frac{1}{p}] \xrightarrow{\text{Galois}} R_{\square}^{\mathrm{cycl}}[\frac{1}{p}]$ $\quad R_\infty[\frac{1}{p}] \xrightarrow{\text{Galois}} R[\frac{1}{p}]$

$R_n :=$ integral closure of $R$ in $(R, R_{\square,n}^{\mathrm{cycl}}) \subset \overline{R}[\frac{1}{p}]$

$$R_\infty = \bigcup_n R_n$$

$$\Gamma_R = \mathrm{Gal}\left( \frac{R_\infty[\frac{1}{p}]}{R[\frac{1}{p}]} \right).$$

$$1 \longrightarrow \Gamma_R' \longrightarrow \Gamma_R \longrightarrow \Gamma_K \longrightarrow 1$$

$\Gamma_R' = \mathrm{Gal}\left( \frac{R_\infty[\frac{1}{p}]}{K_\infty R[\frac{1}{p}]} \right) \cong \mathbb{Z}_p^d.$

$\Gamma_K = \mathrm{Gal}(K_\infty / K) \cong 1 + p^{i(K)}\mathbb{Z}_p$

$$\iota_{\mathrm{cycl}} : R^+_{\mathcal{E}-1,\square} \longrightarrow A^+_{R\square}$$
$$T \longmapsto \pi_i := \varphi^{-i}(\pi).$$
$$X_i \longmapsto [X_i^b]$$

where $\pi = [\varepsilon] - 1 \in A^+_K$.

$\exists! \rightsquigarrow \iota_{\mathrm{cycl}} : R^+_{\varpi} \longrightarrow A^+_R [\![ \frac{p}{\pi_i^{2\cdot 8k}} ]\!]$.

Def: $A_R^{\mathrm{deco}} := \iota_{\mathrm{cycl}}(R_{\varpi}^{\mathrm{deco}})$

$A_{R,\square}^{\mathrm{deco}} := \iota_{\mathrm{cycl}}(R_{\mathcal{E}-1,\square}^{\mathrm{deco}})$.

$G_R \curvearrowright A^+_{R,\square}$ factor through $P_R = \langle \gamma_j : 0 \le j \le d \rangle$

$c = \exp(p^i)$.

$$\begin{cases} \gamma_0(\pi_i) = (1+\pi_i)^c - 1 \\ \gamma_j(\pi_i) = \pi_i \quad \forall\ 1 \le j \le d. \\ \gamma_k([X_k^b]) = [\varepsilon][X_k^b] = (1+\pi)\cdot[X_k^b] \\ \gamma_j([X_k^b]) = [X_k^b] \quad \forall\ j \ne k \quad 1 \le k \le d. \end{cases}$$

The rings $A_R.$, $A_R^{(0,v]+}$, $A_R^{(0,v]}$ & $A_R^{[u,v]}$ are stable under $G_R$ which factors through $P_R$.

⚠ The action of $P_R$ on $A_R^{[u,v]}$ is analytic.

$\rightsquigarrow \mathrm{Lie}\, P_R \quad \nabla_j := \log \gamma_j = t \cdot \partial_j.$ if $0 \le j \le d$.

where $\partial_j := \partial_{\mathrm{cycl},j}$ on $R_{\varpi}^{[u,v]}[\frac{1}{p}]$

$$= \begin{cases} (1+T)\cdot\frac{\partial}{\partial T} & j=0 \\ X_i \frac{\partial}{\partial X_i} & 1\le j\le d. \end{cases}$$

$$Cycl(R_{\varpi}^{[u,v]}, r) \cong Kos(\varphi, \partial, F^r R_{\varpi}^{[u,v]}) = Kos(\varphi, \partial, F^r A_R^{[u,v]})$$

(5). From $(\varphi,\partial)$-module to $(\varphi,\Gamma)$-module.

$$Kos(\varphi,\partial, F^r A_R^{[u,v]}) = \left[\begin{array}{ccc} Kos(\partial', F^r A_R^{[u,v]}) & \xrightarrow{p^r - p^{\bullet}\varphi_{\partial'}} & Kos(\partial', A_R^{[u,\frac{v}{p}]}) \\ \downarrow \partial_0 & & \downarrow \partial_0 \\ Kos(\partial', F^r A_R^{[u,v]}) & \xrightarrow{p^r - p^{\bullet+1}\varphi_{\partial'}} & Kos(\partial', A_R^{[u,\frac{v}{p}]}) \end{array}\right]$$

$\Gamma_R' \cong \mathbb{Z}_p^d$ let $\tau_j = \gamma_j - 1$ $1\le j\le d$.

then $\mathbb{Z}_p[\![\Gamma_R']\!] \cong \mathbb{Z}_p[\![\tau_1, \dots, \tau_d]\!]$.

the usual Koszul complex associated to the regular sequence $(\tau_1, \dots, \tau_d)$.

is $K(\tau_1,\dots,\tau_d) := 0 \to \mathbb{Z}_p[\![\Gamma_R']\!]^{J_d'} \xrightarrow{d_{d-1}'} \cdots \xrightarrow{d_1'} \mathbb{Z}_p[\![\Gamma_R']\!]^{J_1'}$
$\downarrow d_0'$
$\mathbb{Z}_p[\![\Gamma_R']\!]^{J_0'}$
$\downarrow$
$0$

when $d_q'(a_{i_1\cdots i_q}) = \sum_{i=1}^{q} (-1)^{k+1} \cdot a_{i_1 \cdots \hat{i_k} \cdots i_q} \cdot \tau_{i_k}$.

Gives a resolution of $\mathbb{Z}_p = \frac{\mathbb{Z}_p[\![\Gamma_R']\!]}{(\tau_1,\dots,\tau_d)}$ in the cat. of top. $\mathbb{Z}_p[\![\Gamma_R']\!]$-modules.

$K(\tau_1^c, \dots, \tau_d^c)$

$\Lambda := \mathbb{Z}_p[\![\mathbb{P}_K]\!]$

Def: $K(\Lambda) := \quad 0 \to \Lambda^{J_d'} \xrightarrow{d_{d-1}'} \cdots \xrightarrow{d_1'} \Lambda^{J_1'} \xrightarrow{d_0'} \Lambda^{J_0'} \to 0$

(1) $\quad \cong$

$\left(\varprojlim_m \mathbb{Z}_p\left[\mathbb{P}_K / \mathbb{P}_K^{p^m}\right] \otimes_{\mathbb{Z}_p} K(\tau_1, \cdots, \tau_d)\right.$

as left $\Lambda$- & right $\mathbb{Z}_p[\![\tau_1, \cdots, \tau_d]\!]$-mod.

$\rightsquigarrow$ a resolution of $\mathbb{Z}_p[\![\mathbb{P}_K]\!]$ in the cat. of left $\Lambda$-modules.

(2) $\quad K(\Lambda, \tau) := [K^c(\Lambda) \xrightarrow{T_0} K(\Lambda)]$

when $T_0^q : \Lambda^{J_q'} \longrightarrow \Lambda^{J_q'}$

$$a_{i_1 \cdots i_q} \longmapsto (a_{i_1 \cdots i_q}(\gamma_0 - \delta_{i_1 \cdots i_q}))$$

with $\delta_{i_1 \cdots i_q} = \delta_{i_q} \cdots \delta_{i_1}$

$$= \prod_{j=1}^{q} (\gamma_{i_j}^{c} - 1) \cdot (\gamma_{i_j} - 1)^{-1}$$

resolution of $\mathbb{Z}_p$ in the category of top. left. $\Lambda$ mod.

$\Big( \quad 0 \to \mathbb{Z}_p[\![\mathbb{P}_K]\!] \xrightarrow{\gamma_0 - 1} \mathbb{Z}_p[\![\mathbb{P}_K]\!] \to 0 \quad$ exact $\Big)$

$\forall$ top. $\mathbb{P}_K$-mod. $M$.

$$\mathrm{Kos}(\mathbb{P}_R, M) := \mathrm{Hom}_{\Lambda\text{-cont}}(K(\Lambda,\tau), M).$$

$$= \left[\, \mathrm{Kos}(\mathbb{P}_R', M) \xrightarrow{\tau_0} \mathrm{Kos}^c(\mathbb{P}_R', M) \,\right]$$

where $\mathrm{Kos}^?(\mathbb{P}_R', M) := \mathrm{Hom}_{\Lambda\text{-cont}}(K^?(\Lambda), M)$

$$= \mathrm{Hom}_{\Lambda}(K^?(\Lambda), M).$$

<u>Fact</u>: $\mathrm{Kos}(\mathbb{P}_R, M) \xrightarrow{\sim} R\Gamma_{cont}(\Lambda, M) = R\Gamma_{cont}(\mathbb{P}_R, M)$.

Using the fact $K(\Lambda,\tau)$ & the standard complex

$$X_\bullet := (\Lambda \hat{\otimes} T^n \Lambda)_n$$

are two proj. resolution of $\mathbb{Z}_p$.

<u>Def</u>: $\mathrm{Kos}(\varphi, \mathbb{P}_R, A_R^{[u,v]}(r)) :=$

$$\left[\begin{array}{ccc} \mathrm{Kos}(\mathbb{P}_R', A_R^{[u,v]}(r)) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\mathbb{P}_R', A_R^{[u,v/p]}(r)) \\ \downarrow \tau_0 & & \downarrow \tau_0 \\ \mathrm{Kos}^c(\mathbb{P}_R', \text{---}) & \xrightarrow{1-\varphi} & \mathrm{Kos}^c(\text{---}) \end{array}\right]$$

$$= \left[\, \mathrm{Kos}(\mathbb{P}_R, M) \xrightarrow{1-\varphi} \mathrm{Kos}(\mathbb{P}_R, M) \,\right]$$

**Prop 4.2.** $\exists$ a universal constant $N$ & a natural $p^{N\cdot r}$ q-iso

$$\underline{\tau_{\leq r}}\, \mathrm{Kos}(\varphi, \Gamma_R, A_R^{[u,v]}(r)) \xrightarrow{\sim} \tau_{\leq r}\, \mathrm{Kos}(\varphi, \partial, F^r A_R^{[u,v]})$$

Sketch: (1) Lazard: pass from Group to Lie alg.

(2) $F^r A_R^{[u,v]}\underline{(r)} \cong t^r \cdot A_R^{[u,v]}$ $\quad$ $p^{2r}$-iso & Galois-equiva.

$$t^r A_R^{[u,v/p]}(r) \xrightarrow{\sim} A_R^{[u,v/p]}$$

(3) Show $\tau_{\leq r}\, \mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, A_R^{[u,v]}(r))$

$$\simeq \tau_{\leq r}\, \mathrm{Kos}(\varphi, \partial, F^r A_R^{[u,v]}).$$

$\square$

(6). change of annulus of convergence & disk of convergence.

**Prop:**

$$\begin{tikzcd} \mathrm{Kos}(\varphi, \Gamma_R, A_R(r)) & \mathrm{Kos}(\varphi, \Gamma_R, A_R^{(0,v]+}(r)) \arrow[l, "p^8\text{-q-iso}"', "(2)"] \arrow[d, "(1)", "\wr\ \text{q-iso}"'] \\ & \mathrm{Kos}(\varphi, \Gamma_R, A_R^{[u,v]}(r)). \end{tikzcd}$$

Sketch: For (1). $1-\varphi : \dfrac{R_{\varpi}^{[u,v]}}{R_{\varpi}^{[0,v]+}} \longrightarrow \dfrac{R_{\varpi}^{[u,v/p]}}{R_{\varpi}^{[0,v/p]+}}$ is an iso.

Let ~~M =~~ $1-\varphi : M \to M$.

$$\varphi(M) \subset p\cdot M.$$

$\star$: check $M$ doesn't have $p$-divisible elements.

For (2) use $\psi$-complex in the $(\varphi,\Gamma)$-module theory.

& $\Big\{$ $\mathrm{Kos}(\psi, \Gamma_R, A_R^{(0,r]+}(r)) \longrightarrow \mathrm{Kos}(\psi, \Gamma_R, A_R(r))$ is inj.

quot. complex is acyclic. □

(7). $(\varphi,\Gamma)$-modules & Galois coh.

$$\begin{tikzcd}
\overline{R} \arrow[r, dash] \arrow[d, dash] \arrow[rr, bend left, dash] & \overline{R}[\tfrac{1}{p}] \arrow[d, dash] & \text{max. ext. s.t. } \dfrac{\overline{R}[\frac{1}{p}]}{\widetilde{R}_\infty[\frac{1}{p}]} \text{ is étale.} \\
\widetilde{R}_\infty \arrow[r] \arrow[d, dash] & \bigcup_m R[\tfrac{1}{p}]\langle \xi_m, X_{a+b+1}^{\frac{1}{m}}, \dots, X_d^{\frac{1}{m}} \rangle & \\
R \arrow[r, hook] & R[\tfrac{1}{p}] &
\end{tikzcd}$$

$$\widetilde{\Gamma}_R = \mathrm{Gal}\left(\frac{\widetilde{R}_\infty[\frac{1}{p}]}{R[\frac{1}{p}]}\right).$$

$$1 \to \underbrace{\boxed{\prod_{l \neq p} (\mathbb{Z}_l(1))^c}}_{\text{prime to } p} \longrightarrow \widetilde{\Gamma}_R \to \Gamma_R \to 1$$

$\widetilde{R}_\infty[\frac{1}{p}]$ is perfectoid, Apply the const. of ring of periods to $\widetilde{R}_\infty[\frac{1}{p}]$

$$E_{\widetilde{R}_\infty} \supset E^+_{\widetilde{R}_\infty} \qquad A_{\widetilde{R}_\infty} = W(E_{\widetilde{R}_\infty}) = (A_{\overline{R}})^{\mathrm{Gal}\left(\frac{\overline{R}[\frac{1}{p}]}{\widetilde{R}_\infty[\frac{1}{p}]}\right)}$$

$\circledast$: the Artin-Schreier exact sequence.

$$0 \to \mathbb{Z}_p \to A_{\bar{R}} \xrightarrow{1-\varphi} A_{\bar{R}} \to 0$$

induces a $q$-iso.

$$R\Gamma_{cont}(G_R, \mathbb{Z}_p(r)) \longrightarrow [R\Gamma_{cont}(G_R, A_{\bar{R}}(r)) \xrightarrow{1-\varphi} R\Gamma_{cont}(G_R, A_{\bar{R}}(r))]$$

$\circledast$: $A_{R_\infty} \subset A_{\tilde{R}_\infty} \subset A_{\bar{R}}$. induces $q$-iso. by $\underline{\text{almost étale des.}}$

$$R\Gamma_{cont}(\Gamma_R, A_{R_\infty}(r)) \xrightarrow{\sim} R\Gamma_{cont}(\tilde{\Gamma}_R, A_{\tilde{R}_\infty}(r)) \xrightarrow{\sim} R\Gamma_{cont}(G_R, A_{\bar{R}}(r))$$

the kernel of $\tilde{\Gamma}_R \to \Gamma_R$ is of order prime to $p$.

$\circledast$: $R\Gamma_{cont}(\Gamma_R, A_R(r)) \xrightarrow{\psi_\infty} R\Gamma_{cont}(\Gamma_R, A_{R_\infty}(r))$

$\underline{\text{Finally}}$: get $\alpha_r^{Laz}$:

$\underline{\text{Thm}}$ (4.1.4). Assume $K$ contains enough roots of unity. $\exists$ a universal cons. $N$ & $c_p$ s.t. $\exists$ $p^{Nr+c_p}$-$q$-iso.

$$\alpha_r^{Laz}: \tau_{\le r} \operatorname{Syn}(R, r) \xrightarrow{\sim} \tau_{\le r} R\Gamma_{cont}(G_R, \mathbb{Z}_p(r))$$

$$\alpha_{r,n}^{Laz}: \tau_{\le r} \operatorname{Syn}(R, r)_n \xrightarrow{\sim} \tau_{\le r} R\Gamma_{cont}(G_R, \mathbb{Z}/p^n(r))$$

$\square$