2022. 9. 26. Sympathetic closures

Λ = Banach $\mathbb{C}$-algebra, Spectral, connected

Want to construct a "minimal" Λ-algebra. $\tilde{\Lambda}$, which is furthermore p-closed.

(called Sympathetic closure)

way to construct it:

$$\Lambda \rightsquigarrow \Lambda^{(1)} \text{ by adding all } p\text{-th roots of } \mathcal{O}_\Lambda^{**}.$$

$$\rightsquigarrow \Lambda^{(n)} \rightsquigarrow \Lambda^{(\infty)} = \bigcup_{n\geq 1} \Lambda^{(n)} \rightsquigarrow \text{complete to get } \tilde{\Lambda}$$

- to study $\Lambda^{(1)}$, we first study elementary extension. Λ'/Λ.

§1. Elementary extension:

let Λ = Spectral normed $\mathbb{C}$-algebra

Λ'/Λ is called elementary if $\Lambda' \simeq \Lambda[X]/(X^p - x)$ for some $x \in \mathcal{O}_\Lambda^{**}$, which does not have p-th root in Λ.

(like. Kummer ext in. field theory) $\Rightarrow$ • elementary extension is ~~étale~~.

Fact: • $\mathrm{Aut}(\Lambda'/\Lambda) \cong \mu_p$

$$\varepsilon \longmapsto \sigma_\varepsilon(X) = \varepsilon \cdot X$$

• $\Lambda'^{\mathrm{Aut}(\Lambda'/\Lambda)} = \Lambda$

• if $\lambda' = \sum_{i=0}^{p-1} \lambda_i X^i \in \Lambda'$, $\sigma_\varepsilon(\lambda') = \sum_{i=0}^{p-1} \lambda_i (\varepsilon X)^i = \sum_{i=0}^{p-1} \varepsilon^i (\lambda_i X^i)$

then $\sum_{i=0}^{p-1} \varepsilon^{-i} \sigma_\varepsilon(\lambda') = p \cdot \lambda_i X^i$

$\Rightarrow (\ast)^p = p^p \lambda_i^p x^i$ (use $X^p = x$)

ie can express λ_i^p in terms of $\sigma_\varepsilon(\lambda')$

Prop: if Λ is a domain, and integrally closed in $L = \mathrm{Frac}\,\Lambda$.

then. so is Λ'.

Pf: • $L' := L[x]/(x^p - x)$ is a field ext of L. (as $x^p - x$ is irreducible. in $\Lambda[x]$
Λ integ closed $\Rightarrow$ irred in $L[x]$.)

$\uparrow$ (use x is invertible)

$\Lambda[x]/(x^p - x)$ so. Λ' is again a domain

• Λ' integral over Λ, (as X is)

It suffices to show Λ' is equal to the integral closure of Λ in L'

If $\lambda' = \sum \lambda_i x^i$ is integral over Λ $(\lambda_i \in L)$ $\Rightarrow$ $\sigma_\varepsilon(\lambda')$ is so $\Rightarrow \lambda_i^p$ is so

$\Rightarrow \lambda_i$ is so $\Rightarrow \lambda_i \in \Lambda \Rightarrow \lambda \in \Lambda'$. $\square$

Prop: If Λ is connected. then so is Λ'.

Pf: let. $e \in \Lambda'$, $e^2 = e$, $e \neq 0$. want to prove $e = 1$. (enough to prove e is invertible in Λ')

for $\varepsilon \in \mu_p$, $\rightsquigarrow \sigma_\varepsilon(e)$. $\rightsquigarrow$ also idempodent $\neq 0$.

choose $I \subseteq \mu_p$ maximal s.t. $f_I := \prod_{\varepsilon \in I} \sigma_\varepsilon(e) \neq 0$.

(1) if $I = \mu_p$. then $f_I \in \Lambda$, and idempotent, non-zero. $\Rightarrow$. $f_I = 1$.

$\Rightarrow e$ is invertible. in Λ'.

(2) if $I \neq \mu_p$, consider $f_\varepsilon := \sigma_\varepsilon(f_I)$ for $\varepsilon \in \mu_p$, again idempotent of Λ'
and $f_\varepsilon \cdot f_{\varepsilon'} = 0$. if $\varepsilon \neq \varepsilon'$

(because $f_1 \cdot f_{\varepsilon^{-1}\varepsilon'} = 0$)

write $f_I = \sum_{i=0}^{p-1} \lambda_i x^i$. then $(*) \Rightarrow$

$$\underline{\left(\sum_{\varepsilon \in \mu_p} \varepsilon^{-i} f_\varepsilon\right)^p} = p^p \lambda_i^p \cancel{x^{i}}$$

$$\| \qquad \sum_\varepsilon f_\varepsilon \qquad \| \qquad p\lambda_0 \quad (\text{again idempotent in } \Lambda)$$

$\varepsilon^p = 1$

f_ε orthogonal. idemp.

(reduced)

If $p\lambda_0 = 0$., then $\lambda_i^p = 0$, $\Rightarrow \lambda_i = 0$

if $p\lambda_0 = 1$. then. $p^p \lambda_i^p x^i = 1$

$\Rightarrow x$ is p-th power, ok

$\square$

Now. define a norm on Λ':

$$\|\lambda'\|_\infty = \sup_{0\le i\le p-1} \|\lambda_i\|_\Lambda$$

this is. $\mathcal{C}$-alg norm. but not the ~~Spectral norm~~. (only equivalent to)

(need $\|\lambda'^n\|_\infty = \|\lambda'\|_\infty^n$)

Prop: Λ'/Λ elementary extension, then $\exists$ unique. norm. $\|\cdot\|_{\Lambda'}$, which extends the one on Λ. and satisfies:

(i). if Λ'' is normed $\mathcal{C}$-alg. then $\varphi: \Lambda' \to \Lambda''$ is continuous. for $\|\cdot\|_{\Lambda'}$ iff $\varphi|_\Lambda$ is continuous

(ii) $\|X^n\|_{\Lambda'} = \|X\|_{\Lambda'}^n$. | (Moreover, this is the spectral norm on Λ'.

Pf: • first check that such a norm. is equivalent to $\|\cdot\|_\infty$.

write $\lambda' = \sum_{i=0}^{p-1} \lambda_i X^i$. as $\|X^p\| = \|X\|^p = \|x\|_\Lambda = 1$. get $\|X\| = 1$. $= \|X^i\|$ $\forall i$

$\Rightarrow \|\lambda'\| \le \|\lambda'\|_\infty$

for the other inequality, the morphism $(\Lambda', \|\cdot\|_{\Lambda'}) \xrightarrow{id} (\Lambda', \|\cdot\|_\infty)$ is continuous. as. on Λ is. so by (i). get $\|\cdot\|_\infty \le C. \|\cdot\|_{\Lambda'}$. OK

As a consequence, two such norms are equal, using 1-Lipschitz property.

$\Rightarrow$ unicity.

• existence. check that the norm

$$\|\lambda\|_{\Lambda'} := \sup_{s\in S(\Lambda')} |s(\lambda)|, \quad \text{where } S(\Lambda') = \{s: \Lambda' \to \mathcal{C} \mid s|_\Lambda \text{ is continuous}\}$$

(non-empty, $\mathcal{C}$ is p-closed)

is equivalent to $\|\cdot\|_\infty$, So satisfies (i).;

also satisfies (ii) clearly. □

Lemma: If Λ' is an elementary ext of Λ, $b\in\Lambda'$ s.t $b^p \in \mathcal{O}_\Lambda^{**}$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\, y$

Then either : $b\in\Lambda$ and $\Lambda[b] = b$

or $b\notin\Lambda$ and. $\varphi_b: \Lambda[Y]/(Y^p - y) \to \Lambda'$ is an isomorphism.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad Y \mapsto b$

Pf: Exercise (analogue to Kummer extension)

§. p-extension.

Λ = $\mathbb{C}$-algebra, spectral and connected

Λ'/Λ = p-extension, if $\exists$ well-ordered set I, and $(\Lambda_i)_{i\in I}$ s.t.

(i). $\Lambda' = \bigcup_{i\in I} \Lambda_i$, $\Lambda_0 = \Lambda$

(ii) if i is a successor point of I, then Λ_i/Λ_j is elementary.

(iii) if i is a limit point, then $\Lambda_i = \bigcup_{j<i} \Lambda_j$.

$\rightsquigarrow$ $(\Lambda_i)_{i\in I}$ called a presentation of Λ.

Ex: • Λ' is also spectral, connected (by transfinite induction)

• $\mathrm{Spec}(\Lambda') = \{ s \in \mathrm{Hom}(\Lambda', \mathbb{C}) \mid s|_\Lambda \text{ is continuous} \}$

• if Λ'' is a sympathetic normed $\mathbb{C}$-algebra, $\psi: \Lambda \to \Lambda''$ continuous, then ψ extends to continuous $\psi': \Lambda' \to \Lambda''$.

• If Λ is integral domain and integrally closed in $\mathrm{Frac}(\Lambda)$, then so is Λ'!

• if $\Lambda \subseteq \Lambda' \subseteq \Lambda''$, both Λ', Λ'' are p-extensions of Λ. Then Λ'' is a p-ext of Λ'.

Prop: let Λ', Λ'' two p-ext of Λ, $\psi: \Lambda' \to \Lambda''$, morphism of Λ-alg. Then ψ is an isometry. $\Lambda' \xrightarrow{\sim} \mathrm{Im}(\psi)$.

Pf: use transfinite induction to reduces to the case $\Lambda' = \Lambda[X]/(X^p - x)$ elementary.

let $b = \psi(X)$, then $\Lambda' \xrightarrow{\sim} \Lambda[b] \underset{\substack{\uparrow \\ \text{p-extension.}}}{\subseteq} \Lambda''$ + | $\mathrm{Spec}(\Lambda'') \to \mathrm{Spec}(\Lambda[b])$ surj (as any $\Lambda[b] \to \mathbb{C}$ extends to $\Lambda'' \to \mathbb{C}$).

given $\lambda' \in \Lambda'$. you compare $\|\lambda'\|_{\Lambda'}$ and $\|\psi(\lambda')\|_{\Lambda''}$.

(spectral norm)

§. p-closure $\Lambda^{(\infty)}$

Λ = normed $\mathbb{C}$-algebra, spectral, connected.

$\Lambda^{(1)} = \Lambda[\text{all } p\text{-th roots of } \mathcal{O}_\Lambda^{**}]$

if $(x_i)_{i\in I(\Lambda)}$, $x_i \in \mathcal{O}_\Lambda^{**}$, which forms a basis of $G(\Lambda) = \mathcal{O}_\Lambda^{**}/(\mathcal{O}_\Lambda^{**})^p$

then $\Lambda^{(1)} = \Lambda[X_i,\ i\in I(\Lambda)]/(X_i^p - x_i)$. use well-ordering theorem.

Fact: $\Lambda^{(1)}$ is a p-extension of Λ $\Rightarrow$ again. spectral. connected.

Lemma: if $\psi: \Lambda^{(1)} \to \Lambda^{(1)}$ is $\mathbb{C}$-alg morphism, s.t $\psi|_\Lambda : \Lambda \to \Lambda$ is an isometry

then ψ itself is an isometry.

Pf: clearly ψ is bijection. as $\psi|_\Lambda$ is;

it is continuous b/c $\psi|_\Lambda$ is.; also its inverse is continuous $\Rightarrow$ isometry (1-Lipschitz)

Then we can iterate the construction to define $\underset{(\Lambda^{(n-1)})^{(1)}}{\overset{\Lambda^{(n)}}{\shortparallel}}$, and finally $\Lambda^{(\infty)} = \bigcup_{n\geq 1} \Lambda^{(n)}$

by construction. $\Lambda^{(\infty)}$ is p-closed.

Thm: let Λ = normed $\mathbb{C}$-alg, spectral and connected.

Then $\Lambda^{(\infty)}$ is the unique p-extension of Λ which is also. p-closed, (up to isometry)

Moreover if $\psi: \Lambda^{(\infty)} \to \Lambda^{(\infty)}$ is a morphism. s.t. $\psi|_\Lambda$ is isometry $\Lambda \to \Lambda$.

then ψ is an isometry.

Pf: Suppose Λ'' is another one.;

inclusion
$$\begin{tikzcd} \Lambda \arrow[r, hook] \arrow[d, hook] & \Lambda^{(\infty)} \\ \Lambda'' \arrow[ur, dashed] & \end{tikzcd}$$
extends to $\psi: \Lambda'' \to \Lambda^{(\infty)}$, which is isometry $\Lambda'' \xrightarrow{\sim} \mathrm{im}(\psi)$

$\Rightarrow$ $\mathrm{im}(\psi)$ is p-extension., and $\Lambda^{(\infty)}/\mathrm{im}(\psi)$ is again p-ext

but $\mathrm{im}(\psi)$ is p-closed. So $\Lambda^{(\infty)} = \mathrm{im}(\psi)$.

of course. $\Lambda^{(\infty)}$ need not be complete! (even if Λ is), so.

$$\tilde{\Lambda} := \text{completion of } \Lambda^{(\infty)}$$

we call it the sympathetic closure of Λ

<u>Prop</u>: (i) $\tilde{\Lambda}$ is sympathetic

(ii) If $\varphi: \Lambda \to \Lambda'$ is morphism between Banach $\mathbb{C}$-algebras,
Assume. Λ is spectral and connected, Λ' is sympathetic.
then $\exists\ \tilde{\varphi}: \tilde{\Lambda} \to \Lambda'$ extends φ.

(iii). If
$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi_1} & \Lambda_1 \\ \varphi_2 \downarrow & & \downarrow \varphi_1' \\ \Lambda_2 & \xrightarrow{\varphi_2'} & \Lambda' \end{array},$$
all. Banach $\mathbb{C}$-alg.
$\Lambda_1, \Lambda_2, \Lambda'$ are symp.

given. $\tilde{\varphi}_1: \Lambda' \to \Lambda_1$ extends φ_1

then $\exists !\ \tilde{\varphi}_2: \Lambda' \to \Lambda_2$ extending φ_2. making the diagram commute.

(iv) If $\varphi: \tilde{\Lambda} \to \tilde{\Lambda}$ is continuous, and $\varphi|_{\Lambda}$ is isometry, (to Λ).
then φ itself is an isometry

Pf (i) by construction, $\Lambda^{(\infty)}$ is Banach, spectral, p-closed (last talk)
(completion of a colimit of Banach space)

Show: $\tilde{\Lambda}$ is connected.

transfinite induction shows that $\Lambda^{(\infty)}$ is connected.

enough to show: $y^p = 1$ has all roots in. $\mathbb{C}$. (not in $\Lambda - \mathbb{C}$).
ie μ_p

Indeed. let $e^2 = e$, $e \neq 0$. idempotent.

then $e^p = e$, $(1-e)^p = 1-e$, $e(1-e) = 0$.

check that if $\varepsilon \neq 1$, $\varepsilon \in \mu_p$. then $\varepsilon e + (1-e)$ is a solution of $y^p = 1$ but not in $\mathbb{C}$.

let $y \in \tilde{\Lambda}$, with $y^p = 1$. $\Rightarrow$ $\|y\|_{\tilde{\Lambda}} = 1$ (Spectral)

take $x \in \mathcal{O}_{\Lambda^{(\infty)}}$ s.t. $\|x - y\|_{\tilde{\Lambda}} \le p^{-2}$.

$$\Rightarrow \|x^p - 1\| = \|x^p - y^p\| \le p^{-3}$$

$x \in \Lambda_x$ for some finite p-ext Λ_x. $\Rightarrow$ the series $\sum_{n \ge 0} \binom{1/p}{n} (x^p - 1)^n$ then converges to a p-th root of x^p. Say $z \in \Lambda_x$, and $\|z-1\|_{\tilde{\Lambda}} \le p^{-2}$

ie. $(z^{-1} x)^p = 1$,

but $\Lambda^{(\infty)}$ is connected. then $z^{-1} x = \varepsilon \in \mu_p$

Claim: $\varepsilon^{-1} y = z(x^{-1} y) = 1$

$$\|\varepsilon^{-1} y - 1\|_{\tilde{\Lambda}} = \|z(x^{-1}y) - 1\|_{\tilde{\Lambda}}$$

$$= \|z(x^{-1}y) - x^{-1}y + x^{-1}y - 1\|_{\tilde{\Lambda}}$$

$$\le \sup(\|z-1\|_{\tilde{\Lambda}}, \|y - x\|_{\tilde{\Lambda}})$$ (other elements are in $\mathcal{O}_{\Lambda}$)

$$\le p^{-2}.$$

now $\forall s \in \mathrm{Spec}(\tilde{\Lambda})$, $\|\underbrace{s(\varepsilon^{-1} y)}_{\text{p-th root of unity in } C} - 1\|_C \le p^{-2}$

So. $s(\varepsilon^{-1} y) = 1$.

(unique element with $\|\alpha - 1\| \le p^{-2}, \alpha^p = 1$)

$\tilde{\Lambda}$ Spectral $\Rightarrow$ $y = \varepsilon$, as required.

(iii) reduced to the case Λ'/Λ is elementary. $\Lambda' = \Lambda[X]/(X^p - x)$.

$$\begin{tikzcd} x \arrow[d, mapsto] & \Lambda \arrow[r, "\psi_1"] \arrow[d, "\psi_2"] & \Lambda_1 \arrow[d, "\psi_1''"] \\ \psi_2(x) & \Lambda_2 \arrow[r, "\psi_2''"] & \Lambda'' \end{tikzcd}$$

We have $\psi_2(x) \in \mathcal{O}_{\Lambda_2}^{**}$, so has a p-th root $y \in \Lambda_2$, $y^p = \psi_2(x)$

Commutativity $\Rightarrow$ $\psi_1'' \circ \psi_1(x) = \psi_2''(\underline{\psi_2(x)})$ (equiv. $\psi_1'' \circ \psi_1(X^p) = \psi_2''(y^p)$.)

given $\psi_1' : \Lambda' \to \Lambda_1$, $\exists\, \varepsilon \in \mu_p$. $\psi_1'' \circ \psi_1'(X) = \varepsilon \cdot \psi_2''(y)$.

$$\begin{tikzcd} \Lambda' \arrow[r, "\psi_1'"] \arrow[d, "\psi_2'"] & \Lambda_1 \arrow[d, "\psi_1''"] \\ \Lambda_2 \arrow[r, "\psi_2''"] & \Lambda'' \end{tikzcd}$$

it suffices to define $\psi_2'(X) := \varepsilon \cdot y$. (and this is the only way to define it). $\square$

§. $\widetilde{\Lambda\{x\}}$ (here Λ = sympathetic.)

we are interested in $T_\Lambda = \{ \underset{\Lambda\text{-alg isometry}}{\tau : \widetilde{\Lambda\{x\}} \to \widetilde{\Lambda\{x\}}} \mid \tau(x) - x \in \mathcal{O}_\Lambda \}$

$$H_\Lambda\{x\} = \mathrm{Aut}(\Lambda\{x\}^{(\infty)}/\Lambda\{x\}) \;\colon< T_\Lambda$$

$$T_\Lambda \to \mathcal{O}_\Lambda$$

$\tau \mapsto \tau(x) - x$ is group morphism: $(\tau\tau')(x) - x$

$$= \tau\tau'(x) - \tau(x) + \tau(x) - x$$

$$= \tau(\underbrace{\tau'(x) - x}_{\tau'(x)-x}) + \tau(x) - x.$$

Prop: we have an exact sequence

$$0 \to H_\Lambda\{x\} \to T_\Lambda \to \mathcal{O}_\Lambda \to 0$$
$$\tau \mapsto \tau(x) - x$$

Pf: if $\tau(x) - x = 0$. i.e. $\tau|_{\Lambda\{x\}} = \mathrm{id}$. so. $\tau \in H_\Lambda\{x\}$

if $\lambda \in \mathcal{O}_\Lambda$, $\psi : \Lambda\{x\} \to \Lambda\{x\}$, $x \mapsto x + \lambda$ defines (unique). isometry morphism.

So we may lift ψ to $\widetilde{\Lambda\{x\}} \to \widetilde{\Lambda\{x\}}$, continuous

it is automatically isometry: because it is an isometry on $\Lambda\{x\}$. by construction. □

choose $s_c : C\{x\} \to C$ s.t. $s_c(x) = 0$, $\widetilde{C\{x\}} \hookrightarrow \Lambda$

extend $s_c|_{\widetilde{C\{x\}}}$ to $s_\Lambda : \widetilde{\Lambda\{x\}} \to \Lambda$, it induces $T_\Lambda \to \mathrm{Spec}_\Lambda(\widetilde{\Lambda\{x\}}) := \mathrm{Hom}_\Lambda(\widetilde{\Lambda\{x\}}, \Lambda)$

$$\tau \mapsto s_\Lambda \circ \tau.$$

Lemma: this is a homeomorphism. $T_\Lambda \xrightarrow{\sim} \mathrm{Spec}_\Lambda(\widetilde{\Lambda\{x\}})$.

Pf: let $s \in \mathrm{Spec}_\Lambda(\widetilde{\Lambda\{x\}})$, $x := s(x) \in \mathcal{O}_\Lambda$. choose lift $\tau_0 \in T_\Lambda$

$$\begin{tikzcd} \tau_0 \arrow[d, mapsto] & T_\Lambda \arrow[d] \\ x & \mathcal{O}_\Lambda \end{tikzcd}$$

$$\begin{tikzcd} \Lambda\{x\} \arrow[r, "\tau_0|_{\Lambda\{x\}}"] \arrow[d, hook, "\text{inclusion}"'] & \widetilde{\Lambda\{x\}} \arrow[d, "s_\Lambda"] \\ \widetilde{\Lambda\{x\}} \arrow[r, "s"] & \Lambda \end{tikzcd}$$

$x \longmapsto x := s(x)$

commute:
$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & \tau_0(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & s_\Lambda(\tau_0(x)) = s_\Lambda(\tau_0(x) - x) = s_\Lambda(x) = x \end{tikzcd}$$

(as $s_\Lambda h = \mathrm{id}$)

check this diagram lifts to

$$\begin{tikzcd}
\widetilde{\Lambda\{x\}} \arrow[r, "\tau_0"] \arrow[d, "\exists!\ \tilde{\imath}"'] \arrow[dr, phantom, "\wr"] & \widetilde{\Lambda\{x\}} \arrow[d, "s_\Lambda"] \\
\widetilde{\Lambda\{x\}} \arrow[r, "s"] & \Lambda
\end{tikzcd}$$

$\Rightarrow \tilde{\imath} \in T_\Lambda$ (unique)

$\Rightarrow s = s_\Lambda \circ (\tau_0 \tilde{\imath}^{-1}).$ $\square$

Cor: for $f \in \widetilde{\Lambda\{x\}}$, $\|f\|_{\widetilde{\Lambda\{x\}}} = \sup_{\tau \in T_\Lambda} \| s_\Lambda \circ \tau(f) \|$

Pf: Lemma $\Rightarrow \sup_{\tau \in T_\Lambda} \| s_\Lambda \circ \tau(f) \|_\Lambda = \sup_{s \in \mathrm{Spec}_\Lambda(\widetilde{\Lambda\{x\}})} \| s(f) \|_\Lambda \overset{\text{Prop 1.5}}{=} \sup_{s \in \mathrm{Spec}(\widetilde{\Lambda\{x\}})} | s(f) |$

Prop: let $S(\widetilde{\Lambda\{x\}}) = \{ \psi \in \mathrm{Hom}(\widetilde{\Lambda\{x\}}, \widetilde{\mathbb{C}\{x\}}) : \psi(x) = x,\ \psi(\Lambda) = \mathbb{C} \}$

then $\|f\|_{\widetilde{\Lambda\{x\}}} = \sup_{\psi \in S(\widetilde{\Lambda\{x\}})} \| \psi(f) \|_{\widetilde{\mathbb{C}\{x\}}}.$