Talk 4: A funny field $\mathcal{C}$.

Goal: define subset $\mathcal{C} \subseteq \hat{\mathcal{C}}$ of $\widehat{\overline{C\{x\}}}$ and show that they are $\mathbb{Q}_p$-alg.

difficulty: define multiplication on $\hat{\mathcal{C}}$.

Recall notation: C = alg. closed complete non-archi ext of $\mathbb{Q}_p$. $\|p\| = p^{-1}$

$$\begin{array}{ccccc} \overline{F} & \supseteq & \overline{C\{x\}} & \subseteq & \widehat{\overline{C\{x\}}} \\ | & & \cup\!| & & \\ F & \supseteq & C\{x\} & & \end{array}$$

§1. Correspondences on C

Def (1) A comp. $\tilde{f}: C \to C$ (abuse notation) is multiplevalued function

i.e $\forall x \in C$, associate $\{\tilde{f}(x)\} \subseteq C$. image of x

for $E \subseteq C$, define $\tilde{f}(E) = \bigcup_{x \in E} \{\tilde{f}(x)\}$.

If $\{\tilde{f}(x)\} = \phi$, say f is defd at x

(2) For a corresp. $\tilde{f}: C \to C$, the graph of $\tilde{f}$.

$$\Gamma_{\tilde{f}} := \{(x, y) \in C \times C \mid y \in \{\tilde{f}(x)\}\}$$

$\exists$ bijection $\{\text{corresp. } \tilde{f} \text{ on } C\} \longleftrightarrow \{\text{Subsets of } C \times C\}$

$$\tilde{f} \longmapsto \Gamma_{\tilde{f}}$$

(3) composition given $\tilde{f}: C \to C$ and $\tilde{g}: C \to C$

composition $\tilde{f} \circ \tilde{g}: C \longrightarrow C$

$$x \longmapsto \tilde{f}(\{\tilde{g}(x)\})$$

composition is associative $(\tilde{f}\circledcirc\tilde{g})\circledcirc\tilde{h}=\tilde{f}\circledcirc(\tilde{g}\circledcirc\tilde{h})$.

(4). Say $\tilde{f}: C\to C$ is additive if $\Gamma_{\tilde{f}}\subseteq C\times C$ is an additive subgp. In particular, $\{\tilde{f}(0)\}\subseteq C$ is a subgp.

Rmk. $\tilde{f}: C\to C$ additive. $x, y\in C$, when f is dfd

(1) $\{\tilde{f}(x)\}=\{\tilde{f}(0)\}+a,\quad \forall a\in\{\tilde{f}(x)\}$

(2) $\{\tilde{f}(x+y)\}=\{\tilde{f}(x)\}+\{\tilde{f}(y)\}$.

§2. Additive functions

Recall in Shizhang's talk. $S_C: \overline{C\{X\}}\longrightarrow C$
$\qquad X\longmapsto 0$

$$\begin{array}{ccccccccc}1 & \longrightarrow & \widetilde{H}_{C\{X\}} & \longrightarrow & \widetilde{T}_C & \longrightarrow & \mathcal{O}_C & \longrightarrow & 1\\ & & \| & & \text{\"{i}} & & & & \\ & & \mathrm{Gal}_F & & \{\tau\in\mathrm{Aut}(\overline{F}/C)\mid x(\tau):=X^{\tau}-X\in\mathcal{O}_C\} & & & & \end{array}$$

$$\tau\longmapsto x(\tau)$$

Notation: $f\in\widehat{\overline{C\{X\}}}$ $\quad f^{\tau}=\tau(f)$

$S_C^{\tau}:=\left(\widehat{\overline{C\{X\}}}\xrightarrow{\tau}\widehat{\overline{C\{X\}}}\xrightarrow{S_C}C\right)$

$f^{\#}:\ \widehat{\mathbb{B}}(0,1)=\mathrm{Spec}\,\widehat{\overline{C\{X\}}}\longrightarrow C$
$\qquad S\longmapsto S(f)$

$$\begin{tikzcd}
\mathrm{Gal}_F \arrow[r, hook] \arrow[dd] & \widetilde{T}_C \arrow[d, "\tau\mapsto S_C^{\tau}"] \arrow[dd, bend right, "\tilde{\pi}"'] \arrow[drr, bend left, "f^{\#}"] & & \\
& \widehat{\mathbb{B}}(0,1) \arrow[d, "\pi"] \arrow[rr, "f^{\#}"] & & C \\
0 \arrow[r, hook] & \mathbb{B}(0,1) \arrow[urr, "f_C"'] & &
\end{tikzcd}$$

$f^{\#}:\ \widetilde{T}_C\longrightarrow C$

$\tau\longmapsto f^{\#}(S_C^{\tau})=S_C(f^{\tau})\longleftarrow$ Colmez's $f(\tau)$

$f(0) = f^{\#}(\mathrm{id}_{\tilde{T}_c}) = S_c(f)$

Define a corresp. $f_{co}: B(0,1) \longrightarrow C$

$$x \longmapsto f^{\#}(\pi^{-1}(x)) = f^{\#}(\tilde{\pi}^{-1}(x))$$

Call $f \in \widehat{C\{X\}}$ an analytic function and f_{co} an ana. corresp.

$$\Gamma_{f_{co}} = \Gamma_f := \{(x(\tau), f^{\#}(\tau)) \mid \tau \in \tilde{T}_c\}$$

Recall $\|f\|_{sp} = \sup_{\tau \in \tilde{T}_c} \|f^{\#}(\tau)\|$

Hence
$$\begin{tikzcd} B(0,1) \arrow[r, "f_{co}"] \arrow[dr] & C \\ & B(0,\|f\|_{sp}) \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

A basic property of ana. corresp.

Lemma. f_{co} sends cpt to cpt.

Pf. $f_{co} = f^{\#} \circ \pi^{-1}$, π is q. cpt. and $f^{\#}$ is cts.

An ana. fun. f is additive if $f^{\#}: \tilde{T}_c \longrightarrow C$ is gp homo.

In this case $\Gamma_{f_{co}} = \Gamma_f \subseteq C \times C$ is a sub gp.

Hence f_{co} is add. corresp.

Define $\hat{\mathcal{C}} \subseteq \widehat{C\{X\}}$ the subset of add. fun.

$f \in \hat{\mathcal{C}}$, $\{f_{C_0}(0)\} \subseteq C$ cpt subgp, hence a $\mathbb{Z}_p$-module.

Say f is of finite rank if $\{f_{C_0}(0)\}$ is a finite rank $\mathbb{Z}_p$-model

Define $\mathcal{C} \subseteq \hat{\mathcal{C}}$ the subset of finite rank add. fun.

Will focus on $\hat{\mathcal{C}}$.

How to characterize add. fun. ?

lemma. $f \in \widehat{C\{x\}}$, then TFAE

(1) f is additive

(2) $f(0) = 0$ & f_{C_0} is add. coresp.

(3) $f(0) = 0$, and $\exists M \subseteq C$ cpt s.t

$$f_{C_0}(x+y) - f_{C_0}(x) - f_{C_0}(y) \subseteq M, \quad \forall x, y \in B(0,1)$$

(4) $f^{\tau} - f$ is const fun. of value $f^{\tau}(0) = f^{\#}(\tau)$.

Pf. (1) $\Longleftrightarrow$ (2) exercise

(2) $\Longrightarrow$ (3). $f_{C_0}(x+y) - f_{C_0}(x) - f_{C_0}(y)$

$$\overset{f_{C_0} \text{ additive}}{=\!=\!=} f_{C_0}(x) + f_{C_0}(y) - f_{C_0}(x) - f_{C_0}(y)$$

$$= f_{C_0}(0) \subseteq \text{cpt}.$$

interesting

(3) $\Longrightarrow$ (1) Recall a useful lemma

lemma ("cpt im $\Rightarrow$ const lem")

If $f \in \widehat{C\{x\}}$, and $\exists M \subseteq C$ cpt. st. $f^{\#}(\mathcal{T}_C) \subseteq M$, then f is const.

For $\tau \in \tilde{I}_c$, define $g_\tau := f^\tau - f - f^\#(\tau)$.

Then $\forall \sigma \in \tilde{I}_c$, $g_\tau^\#(\sigma) = f^\#(\tau\sigma) - f^\#(\sigma) - f^\#(\tau)$

$$\in f_{co}(\underset{\substack{\| \\ \chi(\tau)+\chi(\sigma)}}{\chi(\tau\sigma)}) - f^\#(\chi(\sigma)) - f^\#(\chi(\tau))$$

$$\subseteq M$$

cpt im $\Rightarrow$ const lemma $\Longrightarrow$ g_τ const of value $g_\tau(0) = 0$.

(1) $\Rightarrow$ (4) is prove above as g_τ is constant. of value 0.

$\Updownarrow$

$f^\tau - f$ is const of value $f^\#(\tau)$

(4) $\Rightarrow$ (1) exercise

Example $\forall c \in C$, $f = cX$ is add.

$f = X^n$, is Not add. if $n \geq 2$.

§.3. An approximation result.

Technical goal : $\hat{\mathcal{E}} \subseteq \widetilde{C\{X\}} := \widehat{C\{X\}^{\cos}}$

$\uparrow$ p-closure of $C\{X\}$

Fix $f \in \hat{\mathcal{E}}$, consider the gp homo

$$\mathrm{Gal}_F \hookrightarrow \tilde{I}_c \xrightarrow{f^\#} C \qquad (*)$$

Write $H_f :=$ kernel of $(*)$. $K_f := \bar{F}^{H_f}$

$\forall \varepsilon > 0$, $H_{f,\varepsilon} :=$ preimage of $B(0, \varepsilon) \subseteq C$

$$\underset{\text{open}}{\subseteq} \mathrm{Gal}_F$$

$K_{f,\varepsilon} := \bar{F}^{H_{f,\varepsilon}}$. $\quad B_{f,\varepsilon} := K_{f,\varepsilon} \cap \overline{C[X]}$ = int. closure of $C[X]$ in $K_{f,\varepsilon}$

$$\begin{tikzcd}
\bar{F} \arrow[d, dash] & \supseteq \overline{C[X]} & \subseteq \widehat{\overline{C[X]}} & \supseteq \hat{\mathcal{C}} \arrow[dll, leftrightarrow, "\text{do approx. here.}"] \\
K_{f,\varepsilon} \arrow[d, dash] & \supseteq B_{f,\varepsilon} \arrow[u, phantom, "\cup|"] & & \\
F & \supseteq C[X] \arrow[u, phantom, "\cup|"] & &
\end{tikzcd}$$

Fact: $\forall \tau \in \hat{T}_c$, $\quad \tau(K_{f,\varepsilon}) = K_{f,\varepsilon}$.

(True, for any $K \subseteq K_f$. Hint: show that $\tau^{-1} H_{f,\varepsilon} \tau = H_{f,\varepsilon}$ using additivity of $f^\#$)

Prop. $f \in \hat{\mathcal{C}}$, $\forall \varepsilon > 0$, $\exists\, f_\varepsilon \in B_{f,\varepsilon}$. s.t. $\|f - f_\varepsilon\|_{sp} \lesssim p^2 \varepsilon$.

Pf. Fact (variant of Ax-Sen-Tate by Colmez)

$\forall f' \in \overline{C[X]} \subseteq \bar{F}$, $\exists\, f_\varepsilon \in K_{f,\varepsilon}$. s.t

$$\|f' - f_\varepsilon\|_{sp} \leq p^2 \Delta_{K_{f,\varepsilon}}(f'). \quad \text{Here } \Delta_{K_{f,\varepsilon}}(f') = \sup_{\sigma \in H_{f,\varepsilon}} \|\sigma(f') - f'\|_{sp}$$

Choose f' such that $\|f - f'\|_{sp} \lesssim \varepsilon$.

(Note $\Delta_{K_{f,\varepsilon}}(f) \lesssim \varepsilon$.) Then have

$$\|f - f_\varepsilon\|_{sp} \leq p^2 \varepsilon \qquad \text{(exercise)}$$

Need to modify f into $B_{f,\varepsilon}$.

Write $f = \frac{b_\varepsilon}{g}$, $b_\varepsilon \in B_{f,\varepsilon}$. $g \in \mathcal{O}_C\{X\}$ of norm 1.

To proceed, need a lemma that we admit

$\boxed{\begin{array}{l}\text{Lemma. } g \in \mathcal{O}_C\{X\}. \text{ norm } 1. \text{ Then } \exists\, \tau \in \tilde{T}_C \text{ s.t.} \\ (g, \tau(g)) = \mathcal{O}_C\{X\}.\end{array}}$

Lemma $\Longrightarrow$ $\exists\, u, v \in \mathcal{O}_C\{X\}$. s.t. $ug + v\tau(g) = 1$

Write $y_\varepsilon = f + \tau(f - f_\varepsilon) = \tau(f_\varepsilon) + f^\#(\tau)$

Write $\tilde{f}_\varepsilon = \underbrace{ug f_\varepsilon}_{\in B_{f,\varepsilon}} + \underbrace{v\tau(g) y_\varepsilon}_{\in B_{f,\varepsilon}}$.

$\boxed{\begin{array}{l}\text{Rmk: Somewhere you} \\ \text{need } \tau(B_{f,\varepsilon}) = B_{f,\varepsilon} \\ \text{as } \tau(K_{f,\varepsilon}) = K_{f,\varepsilon}.\end{array}}$

Can check $\|f - \tilde{f}_\varepsilon\|_{sp} \leq p^2 \varepsilon$.

§4. Technical goal $\hat{e} \subseteq \widehat{C[X]}$.

Prop. $\hat{e} \subseteq \widehat{C[X]}$.

Pf. By approx. result, f is a limit of elets in $\bigcup_{\varepsilon > 0} B_{f,\varepsilon}$.

So enough to show $\bigcup_{\varepsilon > 0} B_{f,\varepsilon} \subseteq C\{X\}^{(\infty)}$.

As $C\{X\}^{(\infty)}$ is inte. closed, enough to show

Lemma $\forall\, \varepsilon > 0$, $K_{f,\varepsilon} \subseteq \mathrm{Frac}(C\{X\}^{(\infty)})$

notation conflict here, sorry!!!

write $K = K_{f,\varepsilon}$

Pf. Enough to show $K = F(\sqrt[p^n]{f})$. for some $f \in \mathcal{O}^{**}_{C\{X\}} := \{ f \in \mathcal{O}^\times_{C\{X\}} \mid \|f - 1\|_{sp} < 1 \}$

small gap here:

Step 1. Write $K = F(\sqrt[p^n]{f})$. for some $f \in F^\times$.

Note Gal_K is finite quotient of $\mathrm{Gal}_{K_f} \underset{\text{cpt subgp}}{\hookrightarrow} C$

$\overset{?}{\Rightarrow}$ Gal_K is p-gp. say $\cong \mathbb{Z}/p^n\mathbb{Z}$.

$K_f.\varepsilon$ is a composite of finite ext of F (which is auto. stable under $\tilde{T}_C$ and whose galois gp is p-gp.

non-can

By Kummer theory, $\Delta := \dfrac{(K^\times)^{p^n} \cap F^\times}{(F^\times)^{p^n}} \cong \mathbb{Z}/p^n\mathbb{Z}$ &

$K = F(\sqrt[p^n]{f})$ for any $f \in (K^\times)^{p^n} \cap F^\times$ whose image in Δ

is a generator. Choose such an f.

Now wish to modify f into $\mathcal{O}^{**}_{C\{X\}}$.

Fact: each element $f \in F^\times$ is of the following form

$$f = \left(\prod_i (X-\alpha_i)^{?}\right) \cdot f_0 \cdot c, \text{ where } \alpha_i \in \mathcal{O}_C, \text{ pairwise distinct.}$$

$? \in \mathbb{Z}$, $f_0 \in \mathcal{O}_C^\times + X\mathfrak{m}_C\{X\}$, and $c \in C^\times$ (of norm $\|f\|_{sp}$).

(See Appendix for the proof of claim).

We call $\prod_? (X-\alpha_i)^?$ the divisor part of f. It is uniquely

determined by f (see Appendix).

Step 3. replace f by some other generator of Δ.

$K \subseteq K_f$. $\forall \tau \in \tilde{T}_C$, $\sigma(K) = K$. $\Longrightarrow$ $\Delta \xrightarrow[\sim]{\tau} \Delta$ isom of gps

$f \longmapsto \tau(f)$.

Hence image of $\tau(f) \in (K^\times)^{p^n} \cap F^\times$ is another generator of Δ. So we have $$K = F(\sqrt[p^n]{f}) = F(\sqrt[p^n]{\tau(f)}).$$

Claim. We can choose $\tau \in \tilde{T}_C$ s.t $\tau(f)$ is of the form

$$\tau(f) = h^{p^n} \cdot f_0', \text{ where } h \in F^\times \text{ and } f_0' \in \mathcal{O}_C^\times + X\mathfrak{m}_C\{X\}.$$

(If so, we are done: h is removable and can replace f_0' by $f_0' / f_0'(0) \in 1 + X\mathfrak{m}_C\{X\} \subseteq \mathcal{O}_{C\{X\}}^{**}$.)

Proof of claim: On the one hand,

$$\tau(f) = \prod_i (X - \alpha_i + x(\tau))^? \cdot f_0', \quad f_0' \in \mathcal{O}_C^\times + X\mathfrak{m}_C\{X\}.$$

On the other hand, $\tau(f) = f^{i_\tau} \cdot g^{p^n}$ for some $1 \le i \le p^n$, $g \in F^\times$

as f is a generator of Δ.

Choose $\tau \in \tilde{T}_C$ s.t $\{\alpha_i\} \cap \{\alpha_i - x(\tau)\} = \phi$. It means that divisor part of f does not involve terms like $(X - \alpha_i)^?$.

Now comparing the divisor parts of two expressions of $\tau(f)$, one finds that $\tau(f) = h^{p^n} \cdot f_0'$, where h is of the form $\prod_j (X - \beta_j)^?$ (in particular, belongs to $F^\times$). $\square$

8.5. Multiplication str. on $\hat{\mathcal{E}}^{\circ} := \hat{\mathcal{E}}^{\|\cdot\| \leq 1}$.

Thm. (1) For any $f, g \in \hat{\mathcal{E}}^{\circ}$, $\exists$ unique $h = f \cdot g \in \hat{\mathcal{E}}^{\circ}$ s.t

$$\Gamma_{h_{\infty}} \subseteq \Gamma_{f_{\infty} \odot g_{\infty}}.$$

(2) With multip. given by "$\cdot$" as in (1), $\hat{\mathcal{E}}^{\circ}$ is a $\mathbb{Z}_p$-alg.

Construction of $h = f \cdot g$.

Recall $\hat{\mathcal{E}}^{\circ} \subseteq \hat{\mathcal{E}} \subseteq \widetilde{C\{X\}}$.

Write $\Lambda = \widehat{C\{X\}}$, sympathetic, can form p-closure $\Lambda\{Y\}^{(\infty)}$

and $\widetilde{\Lambda\{Y\}}$. Choose $S_\Lambda : \widetilde{\Lambda\{Y\}} \longrightarrow \Lambda$.

$$Y \longmapsto 0$$

$$1 \to H_\Lambda \longrightarrow T_\Lambda \longrightarrow \mathcal{O}_\Lambda \longrightarrow 1$$

$$T_\Lambda = \{\tau \in \mathrm{Aut}(\widetilde{\Lambda\{Y\}}/\Lambda) \mid Y^\tau - Y \in \mathcal{O}_\Lambda\}$$

$$\tau \longmapsto Y^\tau - Y.$$

Our g has norm ≤ 1, $g \in \mathcal{O}_\Lambda$. Hence can choose $\tau \in T_\Lambda$, st

$Y^\tau - Y = g$. Consider ring homs

$$\begin{tikzcd}
\hat{\mathcal{E}}^{0} \subseteq \widetilde{\Lambda\{X\}} \cong \widetilde{\Lambda\{Y\}} \arrow[r, hook] \arrow[rrr, bend left, "\beta"] & \widetilde{\Lambda\{Y\}} \arrow[r, "\tau", "\cong"'] & \widetilde{\Lambda\{Y\}} \arrow[r, "S_\Lambda"] & \Lambda = \widehat{C\{X\}}
\end{tikzcd}$$

$$f \longmapsto h$$

$$X \longmapsto Y \longmapsto Y^\tau = Y + g \longmapsto g$$

$h := \beta(h) \in \widehat{\mathbb{C}\{X\}}$.

Q. What are we doing here?

$$X \xmapsto{\ \beta\ } g$$

If $f \in \mathbb{C}[X]$, a poly. then.

$$f = f(X) \xmapsto{\ \beta\ } f(g).$$

In spirit. we are doing composition.

Warning: might be dangerous to think in this way since even $f = X^n$ is not additive as we have seen.

Finally, multiplication on $\hat{\mathcal{E}}$

$\forall f, g \in \hat{\mathcal{E}}$. choose m & n. s.t $p^m f \in \hat{\mathcal{E}}^0$ and $p^n g \in \hat{\mathcal{E}}^0$. then define

$$f \cdot g := p^{-m-n} (p^m f) \cdot (p^n g)$$

Thm. $\mathcal{E}$ and $\hat{\mathcal{E}}$ are $\mathbb{Q}_p$-algebra and have embedding of $\mathbb{Q}_p$-algs

$$\begin{aligned} C &\hookrightarrow \mathcal{E} \hookrightarrow \hat{\mathcal{E}}. \\ c &\longmapsto cX \end{aligned}$$

Will see examples of elements in $\mathcal{E} \setminus C$ in later talks

Appendix

Claim (1) each $f \in F^\times$ admits an expression

$$f = \prod_{? \in \mathbb{Z}} (X-\alpha_i)^{?} \cdot f_0 \cdot c, \quad \begin{cases} \text{where } \alpha_i \in \mathcal{O}_C \text{ pairwise distinct, } f_0 \in \mathcal{O}_C^\times + X\mathfrak{m}_C\{X\}, \\ c \in F^\times \text{ (automatically has norm } \|f\|_{sp}). \end{cases}$$

(2) Call $\prod_{i \in \mathbb{Z}} (X-\alpha_i)^{?}$ the divisor part of f, then the divisor part is unique, ie, these α_i's and their multiplicities are unique.

Pf. First write $f = \frac{g}{h} \cdot c$, s.t. $\|g\|_{sp} = \|h\|_{sp} = 1$ & $c \in C^\times$ of norm $\|f\|_{sp}$.

Then by Weierstrass preparation,

$$\begin{cases} g = \prod_{? \geq 1} (X-\alpha_i)^{?} \cdot g_0, & g_0 \in \mathcal{O}_C^\times + X\mathfrak{m}_C\{X\} \\ h = \prod_{? \geq 1} (X-\alpha_i')^{?} \cdot h_0, & h_0 \in \mathcal{O}_C^\times + X\mathfrak{m}_C\{X\}. \end{cases}$$

↑ recalled below

where the decompositions are unique. Hence we can write

$$f = \prod_{? \in \mathbb{Z}} (X-\alpha_i)^{?} \cdot f_0 \cdot c, \quad \text{w/ } \alpha_i \in \mathcal{O}_C,\ f_0 \in \mathcal{O}_C^\times + X\mathfrak{m}_C\{X\},\ c \in C^\times.$$

Here we require that these α_i's pairwise distinct. Or, better, as a first step, we may write

$$\boxed{f = \frac{g}{h} \cdot c, \quad \text{w/} \begin{cases} \|g\|_{sp} = \|h\|_{sp} = 1, \\ c \in C^\times \ (\text{w/ } \|c\| = \|f\|_{sp}) \ \leftarrow \text{automatic } (*) \\ (g, h) = 1. \end{cases}}$$

The uniquess follows fwm

$\boxed{\begin{array}{l}\text{Weirstrass Preparation (the version we are using above)} \\ f \in \mathcal{O}_C\{x\},\ \text{norm } 1,\ \text{then } \exists \text{ unique } g \in \mathcal{O}_C[x], \\ \text{monic, and unique } h \in \mathcal{O}_C^\times + x\mathfrak{m}_C\{x\} \text{ s.t.} \\ \qquad f = g \cdot h.\end{array}}$