Recall: $\hat{\mathscr{C}} = \{ f \in \widehat{C\{X\}} \mid f \text{ is additive} \}$

$\cup\!|$

$\mathscr{C} = \{ f \in \hat{\mathscr{C}} \mid U_f^0 := \{f_\infty(0)\} = f(\tilde{H}_{C\{X\}}) \text{ is of finite rk}/\mathbb{Z}_p \}$

Thm: ① $\mathscr{C}$ & $\hat{\mathscr{C}}$ are both $\mathbb{Q}_p$-alg's

② We have embedding of $\mathbb{Q}_p$-alg's : $C \hookrightarrow \mathscr{C} \subset \hat{\mathscr{C}}$

$c \mapsto [c] = cX$

Last week, we are reduced to showing:

Thm (1) $\forall f, g \in \hat{\mathscr{C}}^0 = \{ f \in \hat{\mathscr{C}} \mid \|f\|_{sp} \leq 1 \}$, $\exists !\ h = f \circ g \in \widehat{C\{X\}}$ s.t. $h(0) = 0$

& $\Gamma_h \subseteq \Gamma_{f_\infty \odot g_\infty}$. Moreover, $h \in \hat{\mathscr{C}}^0$

(2) With multiplication defined above, $\hat{\mathscr{C}}^0$ is a $\mathbb{Z}_p$-alg.

(Most difficult part:
$\exists\, h \in \widehat{C\{X\}}$ s.t. $\Gamma_h \subseteq \Gamma_{f_\infty \odot g_\infty}$.)

Proof of (1): Recall $\hat{\mathscr{C}} \subseteq \widehat{C\{X\}}$. So we need to prove

Lem: if $f \in \widehat{C\{X\}}$ & $g \in \widehat{C\{X\}}$ s.t. $\|f\|_{sp}, \|g\|_{sp} \leq 1$, then $\exists\, h \in \widehat{C\{X\}}$, $\|h\|_{sp} \leq 1$ such that $\Gamma_h \subseteq \Gamma_{f_\infty \odot g_\infty}$.

Pf: $\Lambda := \widehat{C\{X\}}$, sympathetic

Regard f as element in $\widetilde{C\{Y\}} \subseteq \widetilde{\Lambda\{Y\}}$

Choose $\tau \in T_\Lambda = \{ \sigma \in \mathrm{Aut}_\Lambda(\widetilde{\Lambda\{Y\}}) \mid \gamma(\sigma) := \sigma(Y) - Y \in \mathcal{O}_\Lambda \}$

s.t. $\gamma(\tau) = g$

Define $h := S_\Lambda \circ \tau(f)$

(Recall:
$$\begin{tikzcd} \widetilde{C\{Y\}} \arrow[r, "S_C"] \arrow[d] & C \arrow[d] \\ \widetilde{\Lambda\{Y\}} \arrow[r, "S_\Lambda"'] & \Lambda \end{tikzcd}$$
$S_C(Y) = S_\Lambda(Y) = 0$)

Then $\forall\, s \in \widehat{B(0,1)} = \mathrm{Spec}\,\Lambda$, s.t. $s(X) = x$

Consider: $t : \widetilde{\Lambda\{Y\}} \xrightarrow{\tau} \widetilde{\Lambda\{Y\}} \xrightarrow{S_\Lambda} \Lambda \xrightarrow{s} C$

Then $t(Y) = s \circ S_\Lambda(Y + g) = s(g)$, $t(f) = s(S_\Lambda \circ \tau(f)) = s(h)$

$\rightsquigarrow \Gamma_h \subseteq \Gamma_{f_\infty \odot g_\infty}$. Note: $\|h\|_{sp} \leq 1 \Leftrightarrow \Gamma_h \subseteq B(0,1) \times B(0,1)$. $\boxed{\text{III}}$

Continue the proof : It remains to show h is unique & $\in \widehat{\mathcal{E}}^0$.

- **Exercise** : $f, g \in \widehat{\mathcal{E}} \Rightarrow h(0) = 0$.

- **Uniqueness** : Recall " Cpt - Imply - Const Lem " (CIC Lemma , for short)

  For $h \in \widehat{C\{x\}}$, $h \in C \Leftrightarrow h(\tilde{T}_C)$ is cpt

  $(\text{i.e. } \exists M \text{ cpt}, \forall x \in B(0,1), \{f_{\infty}(x)\} \subseteq M)$.

  Now, if h, h' s.t. $\Gamma_h, \Gamma_{h'} \subseteq \Gamma_{f_\infty \odot g_\infty}$, then $\forall x$

  $$\{h_\infty(x)\} - \{h'_\infty(x)\} \subseteq \{f_\infty \odot g_\infty(x)\} - \{f_\infty \odot g_\infty(x)\} \subseteq \underset{\overset{\uparrow}{\text{cpt}}}{\{f_\infty \odot g_\infty(0)\}}$$

  $$\Rightarrow h - h' = h(0) - h'(0) = 0$$

- **$h \in \widehat{\mathcal{E}}^0$** : $\forall x, y \in B(0,1)$, $\{h_\infty(x+y)\} - \{h_\infty(x)\} - \{h_\infty(y)\}$

  $$\subseteq \{f_\infty \odot g_\infty(x+y)\} - \{f_\infty \odot g_\infty(x)\} - \{f_\infty \odot g_\infty(y)\} \subseteq \{f_\infty \odot g_\infty(0)\}$$

  $$\xRightarrow{\text{Prop 5.2 } + h(0)=0} h \in \widehat{\mathcal{E}}^0.$$

Proof of (2) : Need to show : ① $f \cdot (g \cdot h) = (f \cdot g) \cdot h$ ② $\begin{cases} f \cdot (g_1 + g_2) = f \cdot g_1 + f \cdot g_2 \\ (f_1 + f_2) \cdot g = f_1 \cdot g + f_2 \cdot g \end{cases}$

③ $\forall c \in \mathbb{Z}$, $[c] \cdot f = cf = f \cdot [c]$ (By CIC Lemma)

Rmk : ③ $\Rightarrow [c] \cdot f = cf = f \cdot [c]$, $\forall c \in \mathbb{Z}_p$ As **Fact** : $\begin{array}{c} \widehat{\mathcal{E}}^0 \times \widehat{\mathcal{E}}^0 \to \widehat{\mathcal{E}}^0 \\ (f, g) \mapsto f \cdot g \end{array}$ is cts. ▨

§2. $\widehat{\mathcal{E}}$ is a division alg.

Thm : $C \subseteq \mathcal{E} \subseteq \widehat{\mathcal{E}}$ are embedding of division algebras

Key Prop (§4.) (1) $\forall f \in \widehat{\mathcal{E}}$ with $\|f\|_{sp} = 1$, $\exists !\ g := {}^t f \in \widehat{\mathcal{E}}$

s.t. $\Gamma_g = {}^t\Gamma_f = \{(y, x) \mid (x, y) \in \Gamma_f\}$ & $\|g\|_{sp} = 1$

(2) $V_f^0 := \operatorname{Ker} f_\infty$ & ${}^{\bullet}V_{{}^t f}^0$ are both cpt. & $\exists c_1, c_2 > 0$ st.

$$C_1 \left| (V_f^\circ)^{(n-3)} \right| \leq \left| (V_{{}^tf}^\circ)^{(n)} \right| \leq C_2 \left| (V_f^\circ)^{(n+3)} \right|$$

Here, for any $V \subseteq B(0,1)$, $V^{(n)} := V \big/ V \cap B(0,p^{-n})$

Rmk: Only need to show 1st inequality. The 2nd follows from exchanging f & ${}^tf$.

Fact: V cpt $\Leftrightarrow$ V closed & $V^{(n)}$ finite, $\forall n$ $\left(\Rightarrow V \simeq \varprojlim_n V^{(n)}\right)$

"Key Prop $\Rightarrow$ Thm"

- Need to define f^{-1} for $f \in \widehat{\mathcal{C}}$

By rescaling, WMA $\|f\|_{sp} = 1$.

- Since $\Gamma_1 \subseteq \Gamma_{f_{co} \odot {}^tf_{co}}$, $\Gamma_{{}^tf_{co} \odot f_{co}}$ $\Rightarrow$ ${}^tf \cdot f = 1 = f \cdot {}^tf$. $\Rightarrow f^{-1} = {}^tf$
- Fact: $V \subseteq B(0,1)$ of rk $h \Leftrightarrow 0 < \liminf \dfrac{|V^{(n)}|}{p^{nh}} \leq \limsup \dfrac{|V^{(n)}|}{p^{nh}} < +\infty$.

So (2) implies $f \in \mathcal{C}$ iff ${}^tf \in \mathcal{C}$.

§3. Graph Γ_f $\left(\text{Now, } f \in \widehat{\mathcal{C}} \text{ \& } \|f\|_{sp} = 1 \right)$.

Recall: $\forall g \in \widehat{\mathbb{C}\{x\}}$, $\Gamma_g \subseteq B(0,1) \times B(0, \|g\|_{sp})$ $\left(\Leftrightarrow g(\widetilde{T}_c) \subseteq B(0, \|g\|_{sp})\right)$.

Prop: $f(\widetilde{T}_c) = B(0,1)$

Pf: (1) if $f \in \overline{\mathbb{C}\{x\}}$. Recall: $\forall f \in \overline{\mathbb{C}\{x\}}$, $f(0) = 0$, $\exists$ finitely many $\alpha_1, \dots \alpha_n \in B(0, \|g\|_{sp})$

s.t. $B(0, \|f\|_{sp}) \setminus \bigcup_{i=1}^n B(\alpha_i, \|f\|_{sp}^-) \subseteq g(\widetilde{T}_c)$.

Since $f(\widetilde{T}_c)$ is a group, ~~can find~~ $\forall \alpha \in \bigcup_{i=1}^n B(\alpha_i, 1^-)$

can find β s.t. $\beta, \beta + \alpha \in B(0,1) \setminus \bigcup_{i=1}^n B(\alpha_i, 1^-)$

$\Rightarrow \alpha = (\beta + \alpha) - \beta \in f(\widetilde{T}_c)$.

② By approximation result from last time,

$\exists f_1 \in \overline{\mathbb{C}\{x\}}$, $\|f - f_1\|_{sp} \leq p^{-1}$, $f_1(0) = 0$

$\Rightarrow \|f_1\|_{sp} = 1$ & $\forall T \in \widetilde{T}_c$, $f(T) \equiv f_1(T) \mod B(0, p^{-1})$

$\Rightarrow f = f_1$ as functions $\tilde{T}_C \to B(0,1) \to B(0,1)/B(0,p^{-1})$

$$\begin{tikzcd} \Rightarrow f : \tilde{T}_C \arrow[r] & B(0,1)/B(0,p^{-1}) \arrow[l, dashed, bend left] \end{tikzcd}$$ is surjective.

$\ell :=$ set-theoretic section

devissage

$\Longrightarrow$ $\forall x$, Can construct τ_n s.t $\tau_n \to \tau$ & $f(\tau) = x$. by using ℓ

<u>Hint</u> (Lem 2.13) : $\{\tau_n\} \subseteq \tilde{T}_C$ has a accumulation pt $\Leftrightarrow$ So does $\{x(\tau_n)\}$ ∎

<u>Cor</u> : $B(0,p^{-n}) \subseteq \{f_\infty(B(0,p^{-n}))\}$

<u>Pf</u> : $\forall y \in B(0,p^{-n})$, $\exists \tau \in \tilde{T}_C$, $f(\tau) = \frac{y}{p^n} \Rightarrow y = f(\tau p^n) \in \{f_\infty(p^n x(\tau))\}$ ∎

$/\!/\ \mathrm{Gal}(C\{x\}^{(\infty)}/C\{x\})$.

<u>Lem</u> : $\forall n \geq 1$, $H_n := H_{C\{x\}} \cap f^{-1}(B(0,p^{-n-2}))$, $B_n := (C\{x\}^{(\infty)})^{H_n}$

Then $\exists f_n \in B_n$ s.t (1) $\|f - f_n\|_{sp} \leq p^{-n}$ (2) $f_n(0) = 0$ (3) $F(f_n) = K_n = \mathrm{Frac}(B_n)$

$\Rightarrow \|f_n\|_{sp} = 1$.

<u>Notation</u> : $P_n :=$ min. poly. of $f_n / F = \mathrm{Frac}(C\{x\}) \Rightarrow P_n \in \mathcal{O}_C\{x\}[Y] = \mathcal{O}_C\{x, Y\}$

$\bar{P}_n :=$ reduction of P_n in $k_C[x, Y]$, $s :=$ deg of $\bar{P}_1$ in X.

<u>Fact</u> : P_n is regular in X of deg $s[K_n : K_1]$.

<u>Advantage</u> : <u>Recall</u> : $\forall g \in C\{x\}$ with min. poly $P(x, Y)$

$(x, y) \in \Gamma_g \Leftrightarrow P(x, y) = 0$.

<u>Lem</u> : $\forall n \geq 1$, T.F.A.E ($V_f^0 = \mathrm{Ker} f_\infty$, $U_f^0 = f(\tilde{H}_{C\{x\}}) = \{f_\infty(0)\}$).

(1) $\alpha \in V_f^0 + B(0,p^{-n})$; (2) $\exists \tau \in \tilde{T}_C$ s.t. $\alpha = \alpha(\tau)$ & $f(\tau) \in B(0,p^{-n})$

(3) $\{f_\infty(x)\} \subseteq U_f^0 + B(0,p^{-n})$ (4) $\exists \tau \in \tilde{T}_C$, st $\alpha = \alpha(\tau)$ & $f_n(\tau) \in B(0,p^{-n})$

(5) $\exists y \in B(0,p^{-n})$, $P_n(x, y) = 0$

Pf: (2) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5)

$\|f-f_n\|_{sp} \le p^{-n}$ "Advantage" in ④

(3) $\Rightarrow$ (1): $\forall\, y \in \{f_\infty(x)\}$, $\exists\, \tau \in \tilde{T}_C$ s.t. $x = x(\tau)$, $y = f(\tau)$

As $y \in U_f^0 + B(0,p^{-n}) \Rightarrow \exists\, \sigma_1 \in \hat{H}_{C\{x\}}$ (i.e. $x(\sigma_1)=0$) & $a \in B(0,p^{-n})$, s.t. $y = f(\sigma_1) + a$. Since $B(0,p^{-n}) \subseteq \{f_\infty(B(0,p^{-n}))\}$

$\exists\, \sigma_2$ s.t. $x(\sigma_2) \in B(0,p^{-n})$, $f(\sigma_2) = a$

$\Rightarrow y = f(\sigma_1\sigma_2) = f(\tau) \rightsquigarrow f(\tau\sigma_2^{-1}\sigma_1^{-1}) = 0$ i.e. $x(\tau\sigma_2^{-1}\sigma_1^{-1}) = x(\tau) - x(\sigma_2) \in V_f^0$

$\Rightarrow x = x(\tau) \in V_f^0 + x(\sigma_2) \subseteq V_f^0 + B(0,p^{-n})$. ∎

Cor: $B(0,p^{1-n}) \not\subseteq V_f^0 + B(0,p^{-n})$

Pf. If not, $B(0,p^{1-n}) \subseteq \{f_\infty(B(0,p^{1-n}))\} = \{f_\infty(V_f^0 + B(0,p^{1-n}))\}$

$\subseteq U_f^0 + B(0,p^{-n})$. Impossible as U_f^0 is cpt! ∎

§4. Proof of Key prop

Step 1: V_f^0 is closed & $\forall n$, $\left|(V_f^0)^{(n-1)}\right| \le s[K_n:K_1]$

(i) For $x_n \in V_f^0$ s.t. $x_n \to x$, choose $\tau_n \in \tilde{T}_C$ s.t. $x_n = x(\tau_n)$

$\Rightarrow \{\tau_n\}$ has a limit pt $\tau \Rightarrow x(\tau) = x$ & $f(\tau) = \lim f(\tau_n) = 0$ i.e. $x \in V_f^0$.

(ii) Recall: $P_n(X,Y)$ is regular in X of deg $s[K_n:K_1] =: d$

$\Rightarrow P_n(X, p^nY) = 0$ has d roots in $\overline{C\{Y\}}$, namely $h_{n,1}, \dots, h_{n,d}$

$\Rightarrow \forall\, \tau \in T_{C,Y}$ $\left(= \{\tau \in \mathrm{Aut}(C\{Y\}^{(\infty)}/C) \mid y(\tau) := \tau(Y) - Y \in \mathcal{O}_C\}\right)$

$h_{n,i}(\tau)$'s are d roots of $P_n(X, p^n y(\tau)) = 0$

Recall: $T_{C,Y} \to B(0,1)$, $\tau \mapsto y(\tau)$ is surjective.

Previous Lem $\Rightarrow V_f^0 + B(0,p^{-n}) = \bigcup_i h_{n,i}(T_{C,Y})$.

Put $y_{n,i} := h_{n,i}(0)$, $\rho_{n,i} := \|h_{n,i} - y_{n,i}\|_{sp}$

$\Rightarrow$ $\exists$ finite set $J_i \subseteq B(y_{n,i}, \rho_{n,i})$ s.t. $h_{n,i}(T_{C,Y}) \supseteq B(y_{n,i}, \rho_{n,i}) \setminus \bigcup_{\alpha \in J_i} B(\alpha, \rho_{n,i}^-)$

Assume $\rho = \rho_{n,1} = \sup \rho_{n,i}$ $\Rightarrow$ $B(y_{n,1}, \rho) \setminus \bigcup_{\alpha \in J_1} B(\alpha, \rho^-) \subseteq V_f^\circ + B(0, p^{-n})$

Since $V_f^\circ + B(0, p^{-n})$ is a group, $B(0, \rho) \subseteq V_f^\circ + B(0, p^{-n})$

$\Rightarrow$ $\rho \leq p^{1-n}$

$\uparrow$ Choose $y \in B(y_{n,1}, \rho)$ st $y \in V_f^\circ + B(0, p^{-n})$

$\Rightarrow$ $B(0, \rho) \setminus \bigcup_{\alpha \in J_1} B(\alpha - y, \rho^-) \subseteq V_f^\circ + B(0, p^{-n})$

$\Rightarrow$ $(V_f^\circ)^{(n-1)}$ is a quotient of $V_f^\circ / V_f^\circ \cap B(0, \rho) \simeq \dfrac{V_f^\circ + B(0,\rho)}{B(0,\rho)} \subseteq \{B(y_{n,i}, \rho) \bmod B(0,\rho)\}_{i=1}^{d}$

$\Rightarrow$ $|(V_f^\circ)^{(n-1)}| \leq d = s[K_n : K_1]$.

Step 2: Construction of ^+f

Since $f_n(0) = 0$ $\Rightarrow$ $P_n(0,0) = 0$ $\Rightarrow$ $\exists\, g_n \in \overline{C\{Y\}}$ st. $P_n(g_n, Y) = 0$ & $g_n(0) = 0$.

$\left(\Rightarrow \|g_n\|_{sp} = 1 \right.$
as P_n is regular in X $\left.\right)$

Let $Q(X, Y)$ be min poly of $g_n / C\{Y\}$

$\Rightarrow$ Graph of $g_n := \{(g_n(\tau), y(\tau)) \mid \tau \in \tilde{T}_{C,Y}\} = \{(x,y) \mid Q(x,y) = 0\}$

Now $Q \mid P$ $\Rightarrow$ Graph of $g_n \subseteq \Gamma_{f_n}$

$\Rightarrow$ $\forall\, \tau \in \tilde{T}_{C,Y}$, $\exists\, \sigma_n \in \tilde{T}_C$ s.t. $\begin{cases} g_n(\tau) = x(\sigma_n) \\ y(\tau) = f_n(\sigma_n) \end{cases}$ Put $\sigma_n^{-1} \sigma_{n+1} =: u_n$

$\Rightarrow$ $\begin{cases} f(u_n) = f(\sigma_{n+1}) - f(\sigma_n) \equiv f_{n+1}(\sigma_{n+1}) - f_n(\sigma_n) \bmod p^n \equiv 0 \bmod p^n \\ x(u_n) = x(\sigma_{n+1}) - x(\sigma_n) = g_{n+1}(\tau) - g_n(\tau) \end{cases}$

$\hookrightarrow f(u_n) \in B(0, p^{-n})$

Previous Lem
$\Rightarrow$ $x(u_n) = g_{n+1}(\tau) - g_n(\tau) \in V_f^\circ + B(0, p^{-n})$. Note $(g_{n+1} - g_n)(0) = 0$

(**Lem (Cor 2.16)**: $\forall f \in \widehat{C\{X\}}$ st $f(0) = 0$. if $\exists\, \rho \geq 0$ & S cpt st $f(\tilde{T}_C) \subseteq S + B(0, \rho)$
$\Rightarrow$ $\|f\|_{sp} \leq \rho$)

$\xRightarrow{V_f^\circ \text{ cpt}}$ $\|g_{n+1} - g_n\|_{sp} \leq p^{-n}$ $\Rightarrow$ $g := \lim g_n \in \widehat{C\{Y\}}$ & $\|g\| = 1$

Note $\{x(\sigma_n) = g_n(\tau)\}$ is a Cauchy seq. $\Rightarrow$ $\{\sigma_n\}$ has a limit pt σ

s.t. $g(\tau) = x(\sigma)$ & $y(\tau) = f(\sigma)$.

$\Rightarrow$ Graph of $g = \{(g(\tau), y(\tau)) \mid \tau \in \tilde{T}_{C,\gamma}\} \subseteq \{(x(\sigma), f(\sigma)) \mid \sigma \in \tilde{T}_C\} = \Gamma_f$

Regard g as element in $\widehat{C\{x\}}$, then $\Gamma_g = \{(x(\tau), g(\tau)) \mid \tau \in \tilde{T}_C\} \subseteq {}^t\Gamma_f$

Claim: $\forall \sigma \in \tilde{T}_C$, $\sigma(g) = g + g(\sigma)$ $\left(\Rightarrow g \in \hat{\mathcal{C}}\right)$

Indeed, $h_\sigma := \sigma(g) - g - g(\sigma)$. Then $\forall \tau \in \tilde{T}_C$, $h_\sigma(\tau) = g(\tau\sigma) - g(\tau) - g(\sigma)$

$\Rightarrow (h_\sigma(\tau), 0) = (g(\tau\sigma), x(\tau\sigma)) - (g(\tau), x(\tau)) - (g(\sigma), x(\sigma)) \subseteq \Gamma_f$

$\Rightarrow h_\sigma(\tau) \in V_f^0 \Rightarrow h_\sigma \equiv h_\sigma(0) = 0$ (By CIC Lemma).

Step 3: $\Gamma_g = {}^t\Gamma_f$ & g is unique!

As $g \in \hat{\mathcal{C}}$ & $\|g\|_{sp} = 1$, $\exists h \in \hat{\mathcal{C}}$, $\|h\|_{sp} = 1$ s.t. $\Gamma_h \subseteq {}^t\Gamma_g \subseteq \Gamma_f$

$\Rightarrow (h-f)(\tau) \in V_f^0$, $\forall \tau \in \tilde{T}_C$ $\Rightarrow h - f = 0$ (by CIC)

cpt

Exercise: f is unique! By using $U_f^0 = \{f_{\omega}(0)\}$ is cpt.

Step 4: $|(V_f^0)^{(n-3)}| \leq C|(V_{{}^tf}^0)^{(n)}|$

$(V_{{}^tf}^0)^{(n)} = f(\tilde{H}_{C\{x\}}) \big/ f(\tilde{H}_{C\{x\}} \cap f^{-1}(B(0, p^{-n}))) \simeq H_{C\{x\}} \big/ H_{n-2}$

$\Rightarrow |(V_f^0)^{(n-3)}| \leq s[K_{n-2} : K_1] = s\dfrac{[K_{n-2} : F]}{[K_1 : F]} = \dfrac{s}{[K_1 : F]}|(V_{{}^tf}^0)^{(n)}|$ ∎

Rmk: $\{f \in \mathcal{C} \mid \mathrm{rk}\, f(\tilde{H}_{C\{x\}}) = 0\} = C = \{f \in \hat{\mathcal{C}} \mid f \cdot [c] = [c] \cdot f, \forall c \in C\}$.

§5. Vector Space

Def: A Vector Space is a functor $\mathbb{Y}$: $\{$ Sympathetic alg$/\mathbb{C}$ $\}$ $\to$ $\{\mathbb{Q}_p$ vector spaces$\}$ s.t. $\forall \Lambda$ sym. $\mathbb{Y}(\Lambda) \to \mathrm{Hom}(\mathrm{Spec}(\Lambda), \mathbb{Y}(\mathbb{C}))$ is injective.

Example: (1) $\forall$ fin. dim $\mathbb{Q}_p$ vs A, the functor $\Lambda \mapsto A$ is a VS

(2) $\forall d \geq 1$, $\mathbb{W}^d : \Lambda \mapsto \Lambda^d$ is a VS.

Fact: VS is an abel. category (with Kerf, cokerf, ... being defined point-wisely.) So can talk about exact seq.

Def: For $\mathbb{W} \in$ VS, Say $\mathbb{W}$ is finite dim. if it fits into

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[dr] & & \mathbb{W}^d \arrow[r] & 0 \\
& & \mathbb{Y} \arrow[ur] \arrow[dr] & & \\
0 \arrow[r] & U \arrow[ur] & & \mathbb{W} \arrow[r] & 0
\end{tikzcd}$$

for some V, U fin. dim$/\mathbb{Q}_p$ & $d \geq 0$.

← Call this a representation of $\mathbb{W}$.

In this case, define $\mathrm{Dim}\left(\begin{tikzcd} V \arrow[dr] & & \mathbb{W}^d \\ & \mathbb{Y} \arrow[ur] \arrow[dr] & \\ U \arrow[ur] & & \mathbb{W} \end{tikzcd}\right) := (d, \dim U - \dim V)$

Lem: ~~Let~~ $0 \to A \to \mathbb{W}_1 \to \mathbb{W}_2 \to 0$ be an exact seq. of ~~fin. dim~~ VS & $\dim_{\mathbb{Q}_p} A < +\infty$ then $\mathbb{W}_1$ is of fin dim $\iff$ So does $\mathbb{W}_2$

In this case: $\mathrm{Dim}(\mathbb{W}_1) = \mathrm{Dim}(\mathbb{W}_2) + (0, \dim A)$ (Rely on certain representations of $\mathbb{W}_i$)

Pf: "$\Rightarrow$" Write

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[dr] & & \mathbb{W}^d \arrow[r] & 0 \\
& & \mathbb{Y} \arrow[ur] \arrow[dr] & & \\
0 \arrow[r] & U \arrow[ur] & & \mathbb{W}_1 \arrow[r] & 0
\end{tikzcd}$$

Let $V' := \mathrm{Ker}(\mathbb{Y}(\mathbb{C}) \to \mathbb{W}_2(\mathbb{C}))$

$\Rightarrow 0 \to V \to V' \to A \to 0$ exact.

Claim: $V' = \mathrm{Ker}(\mathbb{Y} \to \mathbb{W}_2)$ i.e. $\forall$ Sym. Λ, $V' = \mathrm{Ker}(\mathbb{Y}(\Lambda) \to \mathbb{W}_2(\Lambda))$

Indeed, By functoriality (Λ is $\mathbb{C}$-alg), $V' \subseteq \mathrm{Ker}(\mathbb{Y}(\Lambda) \to \mathbb{W}_2(\Lambda))$

Then Check the surjection by counting dimensions.

"$\Leftarrow$" Similar!

$\square$