# $\boxed{\S 8}$ Fundamental exact seq

Last time p-adic period rings $\mathbb{B}_{dR}^+$, $\mathbb{B}_{max}^+$, $\mathbb{B}_{st}^+$

will consider $t = \log[\varepsilon] \in A_{max} \subset \mathbb{B}_{max}^+$

$\mathbb{B}_{dR} := \mathbb{B}_{dR}^+[\frac{1}{t}]$, $\mathbb{B}_{max} := \mathbb{B}_{max}^+[\frac{1}{t}]$, $\mathbb{B}_{st} := \mathbb{B}_{st}^+[\frac{1}{t}]$.

Goal $\exists$ exact seq.'s

$$0 \to \mathbb{B}_{max} \to \mathbb{B}_{st} \xrightarrow{N} \mathbb{B}_{st} \to 0. \qquad \text{(SEF 1E)}$$

$$0 \to \mathbb{B}_{max}^{\varphi=1} \to \mathbb{B}_{max} \xrightarrow{\varphi-1} \mathbb{B}_{max} \to 0 \qquad \text{(SEF 2E)}$$

$$0 \to \mathbb{Q}_p \to \mathbb{B}_{max}^{\varphi=1} \to \mathbb{B}_{dR}/\mathbb{B}_{dR}^+ \to 0. \qquad \text{(SEF 3E)}$$

Along the way, will show

**Prop** (1) $\dim (\mathbb{B}_{max}^+)^{\varphi = p^s} = (s, 1)$

(2) $\dim(\mathbb{B}_s) = (s, 0)$

$\searrow$ $A_{inf}[\frac{1}{p}]/(\ker\theta)^s$.

Rmk • Colmez considered more gen'l coeff. $E/\mathbb{Q}_p$ finite.

• $\Lambda$ will denote a sympathetic alg.

## $\boxed{1}$ $\varphi = 1$ pts.

$$R = \varprojlim_{\varphi} A/pA, \quad \varphi : R \to R, \quad x \mapsto x^p \quad x \in R(\Lambda).$$

$$A_{\inf} = W(R).$$

$$\rightsquigarrow \varphi : A_{\inf} \to A_{\inf}$$

$$\varphi\left(\sum_{n=0}^{\infty} p^n [x_n]\right) = \sum_{n=0}^{\infty} p^n [x_n^p], \quad x_n \in R(\Lambda).$$

Lem 1 (1) $R^{\varphi=1} = \mathbb{F}_p$.

(2) $A_{\inf}^{\varphi=1} = \mathbb{Z}_p$.

pf) (2) follows from (1)

(1): Let $x = (x^{(n)}) \in R(\Lambda)$ s.t. $\varphi(x) = x^p = x$

$\Rightarrow (x^{(n)})^p = x^{(n)} \in \mathcal{O}_\Lambda$.

Fact (Exc) $\Lambda$ conn. $\Rightarrow$ any poly. $\geq$ deg 2 / $C$ has all roots in $C$.

$\Rightarrow x^{(n)} = [a] \in W(\mathbb{F}_p)$ for some $a \in \mathbb{F}_p$.

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$$I = \theta^{-1}(pA) = (p, [\tilde{p}]) \subset A_{\inf}.$$

stable under $\varphi$. $\quad A_{\max} \circlearrowleft \varphi$.

$$\left( = \left(A_{\inf}\left[\tfrac{I}{p}\right]\right)^{\wedge} \right)$$

Prop 1 $(A_{max})^{\varphi=1} = \mathbb{Z}_p$.

pf) Let $x \in A_{max}(\Lambda)$ s.t. $\varphi(x) = x$.

$$x = \sum_{n=0}^{\infty} b_n \left(\frac{[\tilde{p}]}{p}\right)^n, \quad b_n \in A_{inf}(\Lambda), \quad b_n \to 0$$

$\forall k \geq 1$,
$$x = \varphi^k(x) = \sum_{n\geq 0} \varphi^k(b_n)\, p^{n(p^k-1)} \left(\frac{[\tilde{p}]}{p}\right)^{p^k n}$$

$$\Rightarrow \quad x - \varphi^k(b_0) \in p^{k-1} A_{max}(\Lambda).$$

$$\underset{\substack{\downarrow \\ (p,[\tilde{p}])\text{-adic}}}{A_{inf}(\Lambda)} \underset{\text{clsd}}{\subset} \underset{\substack{\downarrow \\ p\text{-adic}}}{A_{max}(\Lambda)} \quad \Rightarrow \quad x \in A_{inf}(\Lambda)$$

Apply Lem 1 ▨

## 2 Elements $\xi$ & $\omega$

$h(X) := (X+1)^p - 1$. For $n \geq 1$, $h^{[n]} := \underbrace{h \circ \cdots \circ h}_{n \text{ fold}}$

For $x \in \mathbb{R}$, denote $\{x\} := [x+1] - 1$.

We have $\varphi(\{x\}) = h(\{x\})$.

Choose $\varepsilon = (1, \varepsilon_1, \varepsilon_2, \cdots), \ \varepsilon_n \in \mathcal{O}_C, \ \varepsilon_1 \neq 1, \ \varepsilon_{n+1}^p = \varepsilon_n$.

$u_n := \varepsilon_{n+1} - 1. \quad h(u_{n+1}) = u_n \quad v_p(u_n) = \dfrac{1}{(p-1)p^n}$

$\bar{u}_n \in \mathcal{O}_C/p, \quad u = (\bar{u}_n) \in R$

$$v_R(u) = \lim_{n\to\infty} p^n v_p(u_n) = \frac{1}{p-1}.$$

$$w_n := \varphi^{1-n}(\{u\}) = \varphi^{1-n}([u+1] - 1)$$

<u>Def</u> $\cdot\ w := w_0 = [\varepsilon] - 1.$

$\cdot\ \xi := \dfrac{w_0}{w_1} = 1 + [\varepsilon^{\frac{1}{p}}] + \cdots + [\varepsilon^{\frac{p-1}{p}}].$

$J = \ker(\theta) \subset A_{\inf}$

$$J^{[n]} := \bigcap_{i=0}^{n} \varphi^{-i}(J) \subset A_{\inf}.$$

$$J^{[n]}(\Lambda) = \{x \in A_{\inf}(\Lambda) \mid \theta(\varphi^i(x)) = 0 \quad \forall i \le n\}$$

<u>Prop 2</u> (1) $J^{[n]} = \left(\dfrac{w}{w_{n+1}}\right)$. In part., $J = (\xi)$.

(2) $J^{[\infty]} = (w)$.

pf) $\theta(\xi) = 1 + \varepsilon_1 + \cdots + \varepsilon_1^{p-1} = 0.$

$\bar{\xi} = u^{p-1} \in R \Rightarrow v_R(\bar{\xi}) = 1. \Rightarrow J = (\xi).$

(1) Induct on $n$.

Let $x \in \mathbb{J}^{[n+1]}(\Lambda)$. $\exists x_n \in A_{inf}(\Lambda)$ s.t.

$x = \frac{w}{w_{n+1}} x_n$. $\varphi^{n+1}(x) = \frac{\varphi^{n+1}(w)}{w} \varphi^{n+1}(x_n)$.

$\theta\left(\frac{\varphi^{n+1}(w)}{w}\right) = p^{n+1} \neq 0 \Rightarrow \theta(\varphi^{n+1}(x_n)) = 0$

$\Rightarrow \varphi^{n+1}(x_n) \in \frac{w}{w_1} A_{inf}(\Lambda)$. $\Rightarrow \varphi^{n+1}(x_n) = \frac{w}{w_1}\varphi^{n+1}(x_{n+1})$

for some $x_{n+1} \in A_{inf}(\Lambda)$.

$\Rightarrow x = \frac{w}{w_{n+1}} \cdot \frac{w_{n+1}}{w_{n+2}} x_{n+1}$

(2) $w \in \mathbb{J}^{[\infty]}$

<u>Claim</u> $w A_{inf}(\Lambda) \hookrightarrow \mathbb{J}^{[\infty]}$ is surj.

Since both are $p$-adic compl., suffices to check mod $p$.

Let $x \in \mathbb{J}^{[\infty]}(\Lambda) = \bigcap_{n=1}^{\infty} \mathbb{J}^{[n]}(\Lambda)$, $\bar{x} \in R(\Lambda)$.

$v_R(\bar{x}) \geq v_R\left(\frac{\bar{w}}{\bar{w}_n}\right) = (1 - \frac{1}{p^n}) v_R(\bar{w})$. $\forall n$.

$\Rightarrow v_R(\bar{x}) \geq v_R(\bar{w})$

$\Rightarrow \exists\, \bar{y} \in \mathbb{R}(\Lambda)$ s.t. $\bar{x} = \bar{w}\,\bar{y}$.

$z := x - w[\bar{y}] \in \mathbb{J}^{[n]} \cap p\mathbb{A}_{\mathrm{inf}} = p\mathbb{J}^{[n]} \quad \forall n$

$\Rightarrow z \in p\,\mathbb{J}^{[\infty]}$

$\boxed{Q.E.D.}$

## $\boxed{3}$ Element $t$

For $k \geq 1$, $\ell_p(k) := \lfloor \log_p k \rfloor$.

$\log(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \cdots \in \mathbb{Q}_p[\![X]\!]$

$v_p\left(\frac{1}{k}\right) \geq -\ell_p(k) \cdot \frac{1}{p-1}$. $\quad F(X) := \log(1+X)$.

$F \circ h = pF$.

**Def** $t := F(w) = \log(1+w)$. $\left( = \log[\varepsilon] \right)$.

**Prop 3** (1) $t \in A_{\max}$, $\varphi(t) = pt$ & $\frac{t}{w}$ is a unit in $A_{\max}$

(2) $\mathbb{J}^{[\infty]}_{\max} := \{ x \in \mathbb{B}^+_{\max} \mid \theta(\varphi^n(x)) = 0 \quad \forall n \}$

$= t\,\mathbb{B}^+_{\max}$.

pf) (1) $\varphi(t) = \varphi(F(w)) = F(\varphi(w)) = F(h(w))$

$$= pF(w) = pt$$

$$\frac{t}{w} = 1 + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} w^{k-1} \in 1 + pA_{max}.$$

$$\left( k \geq 2 : \begin{cases} p \neq 2 & : \ l_p(k) \leq k-2 \quad \& \ w \in J \\ p = 2 & : \ l_p(k) \leq k-1 \quad \underline{\text{Exc}} \ w \in I^2. \end{cases} \right)$$

(2) <u>claim</u> $J_{max}^{[\infty]} = w B_{max}^{+}$

Let $x \in J_{max}^{[\infty]}(\Lambda)$. Since $\Theta(x) = 0$,

$$x = \sum_{n=1}^{\infty} a_n \left(\frac{\xi}{p}\right)^n = \frac{\xi}{p} \left( \sum_{n=1}^{\infty} a_n \left(\frac{\xi}{p}\right)^{n-1} \right)$$

for some $a_n \in A_{inf}(\Lambda)[\frac{1}{p}]$, $a_n \to 0$.

Set $Q_n(x) = \frac{1}{x}((1+x)^n - 1)$ for $n \geq 1$.

$$\varphi\left(\frac{\xi}{p}\right) = \frac{1}{p} Q_p(w) = 1 + wA(w)$$

$$\varphi(x) = \varphi\left(\frac{\xi}{p}\right) \left( \sum_{n=1}^{\infty} \varphi(a_n) + w\varphi(y) \right) \quad \text{where}$$

$$y = A(w_1) \left( \sum_{n=2}^{\infty} a_n Q_{n-1} \Big( \underbrace{\tfrac{1}{p} Q_p(w_1)}_{\substack{\cap \\ A_{max}}} - 1 \Big) \right) \in B_{max}^{+}(\Lambda)$$

$A_{max}$ b/c $w_1^{p-1} \in I$ (<u>Exc</u>).

$\theta(\varphi^k(x)) = 0 \quad \forall k$

$\Rightarrow \theta(\varphi^k(\sum_{n=1}^{\infty} a_n)) = 0 \quad \forall k \geq 1$

$( \because \theta(\varphi^{k-1}(w)) = 0 \quad \forall k \geq 1 )$

<u>Prop 2(2)</u> $\Rightarrow \varphi(\sum_{n=1}^{\infty} a_n) = wb$ for some

$b \in A_{inf}(\Lambda)[\frac{1}{p}]$

$\Rightarrow x = \frac{\xi w_1}{p}(y + \varphi^{-1}(b)) = \frac{w}{p}(y + \varphi^{-1}(b))$

$\boxed{///}$

## $\boxed{4}$ <u>Vector Sp. $U_s$</u>

$\mathbb{R}(\Lambda)^{**} = \{ x \in \mathbb{R}(\Lambda) \mid \|x-1\|_{\mathbb{R}} < 1 \}$

<u>Lem 2</u> For $x \in \mathbb{R}(\Lambda)^{**}$, $\log[x] := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}([x]-1)^n \in \mathbb{B}^+_{max}(\Lambda)$

Furthermore, $\mathbb{R}(\Lambda)^{**} \longrightarrow \mathbb{B}^+_{max}(\Lambda)$, $x \longmapsto \log[x]$

is a gp hom., $\varphi(\log[x]) = p\log[x]$.

pf) Let $r \geq 1$ s.t. $\|(x-1)^r\|_{\mathbb{R}} \leq \frac{1}{p}$.

$\Rightarrow ([x]-1)^r \in \mathbb{I}(\Lambda)$

$$\log[x] = \sum_{i=1}^{n} ([x]-1)^i \left( \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^{kr+i-1}}{kr+i}}_{\to 0 \text{ as } k\to\infty} p^k \left( \frac{([x]-1)^r}{p} \right)^k \right)$$

$$\in B_{max}^{+}(\Lambda).$$

$\square$

<u>Def</u> $U_s := (B_{max}^{+})^{\varphi=p^s}$. $p$-adic Banach sp.

<u>Rmk</u> $U_0 = (B_{max}^{+})^{\varphi=1} = \mathbb{Q}_p$ (<u>Prop 1</u>)

<u>Prop 4</u> $U_1$ is Vec. sp. of dim $(1,1)$ having present'n.

$$0 \to \mathbb{Q}_p \cdot t \longrightarrow U_1 \xrightarrow{\theta} V' \to 0.$$

pf) For $x \in R(\Lambda)^{**}$, $\theta(\log[x]) = \log x^{(0)}$

$\Rightarrow \theta(t) = \log 1 = 0$ & $t \in U_1 \cap \ker\theta$

Let $x \in U_1(\Lambda)$ s.t. $\theta(x) = 0$.

$\Rightarrow \theta(\varphi^k(x)) = \theta(p^k x) = 0 \quad \forall k \geq 1.$

<u>Prop 3(2)</u> $\Longrightarrow$ $x = ty$ for some

$y \in (B_{max}^{+})^{\varphi=1}(\Lambda) = \mathbb{Q}_p.$

Let $\lambda \in V'(\Lambda) = \Lambda$. Let $n \geq 0$ s.t. $p^n\lambda \in p^2\mathcal{O}_\Lambda$.

$\Rightarrow e^{p^n\lambda} \in 1 + p\mathcal{O}_\Lambda \subset \mathcal{O}_\Lambda^{**}$

$\Lambda$ p-closed $\Rightarrow$ $\exists x = (x^{(m)}) \in R(\Lambda)^{**}$ s.t.

$x^{(0)} = e^{p^n \lambda}$.

$\log[x] \in U_1(\Lambda)$ & $\theta(\log[x]) = \log x^{(0)} = p^n \lambda$. $\boxed{\text{Q.E.D.}}$

<u>Lem 3</u> Let $y \in U_s(\Lambda)$ s.t. $\theta(y) = 0$.

Then $\exists !\, x \in U_{s-1}(\Lambda)$ s.t. $y = tx$.

pf) Follows by similar argument as in <u>pf</u> of <u>Prop 4</u>

Note $t \in B^+_{max}$ is non-zero divisor by:

<u>Fact</u> (1) $B^+_{max} \longrightarrow B^+_{dR}$ injective

(2) $t \in B^+_{dR}$ is uniformizer. $\boxed{\text{Q.E.D.}}$

<u>Prop 5</u> Let $u \in U_1(C)$ be a lift of $1 \in C$

along $\theta$. We have SES

$$0 \to \mathbb{Q}_p \xrightarrow{\times (u^{s-1}, t)} U_{s-1} \oplus U_1 \xrightarrow{(x,y) \mapsto tx - u^{s-1}y} U_s \to 0$$

pf) Let $(x, y) \in U_{s-1}(\Lambda) \oplus U_1(\Lambda)$ s.t. $u^{s-1}y = tx$.

$\theta(y) = 0 \Rightarrow \exists !\, a \in U_0(\Lambda) = \mathbb{Q}_p$ s.t. $y = ta$

( <u>Lem 3</u> )

$\Rightarrow x = u^{s-1} a$, so exactness in the middle.

Let $z \in U_s(\Lambda)$, Prop 4 $\Rightarrow$ $\exists\, y \in U_1(\Lambda)$

s.t. $\theta(y) = -\theta(z)$

$z' := z + u^{s-1}y \in U_s(\Lambda)$. $\theta(z') = 0$.

$\Rightarrow \exists!\, x \in U_{s-1}(\Lambda)$ s.t. $z' = tx$ (Lem 3)

$\Rightarrow z = tx - u^{s-1}y$

$\blacksquare$

Cor 1 $U_s$ is Vec. sp. of dim $(s, 1)$.

pf) Induct on $s$ $s = 1$ : Prop 4

$U_{s-1} \oplus U_1$ : dim $(s, 2)$.

$\mathbb{Q}_p$ : dim $(0, 1)$.

$\Rightarrow U_s$ has dim $(s, 2-1) = (s, 1)$. by Prop 5

Fact Let $0 \to A \to W_1 \to W_2 \to 0$ be

SES of vec. sp's, where $A$ is fin.

dim'l $\mathbb{Q}_p$-VS. & $\dim W_1 = (d, a)$.

Then $\dim W_2 = (d, a - \dim_{\mathbb{Q}_p} A)$.

Rmk For $U_{h,s} := (B^+_{\max})^{\varphi^h = p^s}$, can show

$\dim U_{h,s} = (s, h)$ by considering $E/\mathbb{Q}_p$ unram. deg $h$.

## $\boxed{5}$ Vector sp. $\mathbb{B}_s$

$$\mathbb{B}_s = A_{\mathrm{inf}}[\tfrac{1}{p}] / (\ker\theta)^s .$$

**Prop 6** Nat'l map $\mathbb{B}^+_{\max} \to \mathbb{B}^+_{dR}$ induces SES of Top. vec. sp's

$$0 \to \mathbb{Q}_p \xrightarrow{\times t^s} \mathbb{U}_s \to \mathbb{B}_s \to 0$$

pf) Induct on $s$. $s=1$ : Prop 4

Consider commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Q}_p \arrow[r, "\times t^s"] \arrow[d, equal] & \mathbb{U}_s \arrow[r] \arrow[d, "\times t"] & \mathbb{B}_s \arrow[r] \arrow[d, "\times t"] & 0 \\
0 \arrow[r] & \mathbb{Q}_p \arrow[r, "\times t^{s+1}"] & \mathbb{U}_{s+1} \arrow[r] \arrow[d] & \mathbb{B}_{s+1} \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{V}^1 \arrow[r, equal] \arrow[d] & \mathbb{V}^1 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

<u>Claim</u> col's are exact.

- $\mathbb{U}_{s+1} \to \mathbb{V}'$ surj. :

Let $\lambda \in \mathbb{V}'(\Lambda)$.

$\exists\, x, y \in \mathbb{U}_1(\Lambda)$ s.t. $\theta(x) = 1$, $\theta(y) = \lambda$ (<u>Prop 4</u>)

$\Rightarrow \theta(x^s y) = \lambda$ & $x^s y \in \mathbb{U}_{s+1}(\Lambda)$.

+ Similar argument as in <u>pf</u> of <u>prop 4</u>

$\Rightarrow$ left col. is exact.

$t \in B^+_{dR}$ is a uniformizer

$\Rightarrow$ right col. is exact.

<u>Exc</u> exactness of 1st row & 2 col's

$\Rightarrow$ 2nd row is exact.

$\boxed{\text{Q.E.D.}}$

<u>Cor 2</u> $\dim \mathbb{B}_s = (s, 0)$

pf) <u>Prop 6</u> + $\dim \mathbb{U}_s = (s, 1)$. (<u>Cor 1</u>)

$\boxed{\text{Q.E.D.}}$

## 6 Fund. exact seq.

$\mathbb{B}_{dR} := \mathbb{B}_{dR}^{+}[\frac{1}{t}]$, $\quad \mathbb{B}_{max} = \mathbb{B}_{max}^{+}[\frac{1}{t}]$, $\quad \mathbb{B}_{st} = \mathbb{B}_{st}^{+}[\frac{1}{t}]$.

$\mathbb{B}_{max} \circlearrowleft \varphi$, $\quad \mathbb{B}_{st} \circlearrowleft \varphi, N$.

Thm We have SES's

$$0 \to \mathbb{B}_{max} \to \mathbb{B}_{st} \xrightarrow{N} \mathbb{B}_{st} \to 0 \qquad (1)$$

$$0 \to \mathbb{B}_{max}^{\varphi=1} \to \mathbb{B}_{max} \xrightarrow{\varphi-1} \mathbb{B}_{max} \to 0 \qquad (2)$$

$$0 \to \mathbb{Q}_p \to \mathbb{B}_{max}^{\varphi=1} \to \mathbb{B}_{dR}/\mathbb{B}_{dR}^{+} \to 0 \qquad (3)$$

pf) (1) : follows from $\mathbb{B}_{st}^{+} = \mathbb{B}_{max}^{+}[u]$ where $N$ is $\mathbb{B}_{max}^{+}$-lin. derivation $-\frac{d}{du}$.

(3) : $\mathbb{B}_{max}^{\varphi=1} = \bigcup_{n=1}^{\infty} (t^{-n}\mathbb{B}_{max}^{+})^{\varphi=1}$

Prop 6 $\Rightarrow$ exact seq.

$$0 \to \mathbb{Q}_p \to (t^{-n}\mathbb{B}_{max}^{+})^{\varphi=1} \to t^{-n}\mathbb{B}_{dR}^{+}/\mathbb{B}_{dR}^{+} \to 0$$

$\lim\limits_{n\to\infty} \Rightarrow$ (3).

(2) : Need to show $\varphi - 1 : \mathbb{B}_{max} \to \mathbb{B}_{max}$ is surj.

For $x, y \in B^+_{max}$, we have

$$(1-\varphi)\left(\frac{x}{t^i}\right) = \frac{y}{t^i} \iff \left(1-\frac{\varphi}{p^i}\right)(x) = y.$$

Claim $1-\frac{\varphi}{p^i} : B^+_{max} \longrightarrow B^+_{max}$ is surj.

$$y = \sum_{n=0}^{\infty} a_n \left(\frac{[\tilde{p}]}{p}\right)^n, \quad a_n \in A_{inf}[\tfrac{1}{p}], \quad a_n \to 0.$$

WLOG assume $a_n \in A_{inf}$ $\forall n$.

Idea Consider 2 formal inverses of $1-\frac{\varphi}{p^i}$:

$$\sum_{k=0}^{\infty} p^{-ki}\varphi^k \quad \& \quad -\sum_{k=1}^{\infty} p^{ki}\varphi^{-k}.$$

$$x_0 := -\sum_{k=1}^{\infty} p^{ki}\varphi^{-k}(a_0) \in A_{inf} \quad \&$$

$$(1-p^{-i}\varphi)(x_0) = a_0$$

$$p^{-ki}\varphi^k\left(\left(\frac{[\tilde{p}]}{p}\right)^n\right) = p^{np^k-n-ki}\left(\frac{[\tilde{p}]}{p}\right)^{np^k}$$

$\exists\, m(i)$ s.t. $np^k - n - ki \geq -m(i)$ $\forall n, k \geq 1$.

& $np^k - n - ki \longrightarrow +\infty$ as $n+k \to +\infty$.

$$\Rightarrow \quad x_1 := \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \varphi^k(a_n)\, p^{np^k - n - ki} \left( \frac{[\tilde{p}]}{p} \right)^{np^k} \in p^{-m(i)} A_{max}$$

$$\& \quad (1 - p^{-i}\varphi)(x_1) = y - a_0 .$$

$$\Rightarrow \quad (1 - p^{-i}\varphi)(x_0 + x_1) = y$$

$\square$