2023 Winter – Seminar on *Formal and Rigid Geometry*

Talk 1-2

TATE ALGEBRAS

This is the live-TeXed notes by Wenhan Dai for this seminar. The note-taker claims no originality and takes full responsibility for all errors made therein.

- **Talk 1 (Tate Algebra I):** Cover [Bos14, pp. 9-20] (until Corollary 13). Discuss the definition of restricted power series T_n and relevant properties. In particular, prove the maximum principle (Proposition 5), and then prove the theorem of Weierstrass Division (Theorem 8) and discuss its several corollaries (Corollary 9-13).
- **Talk 2 (Tate Algebra II):** Cover [Bos14, pp. 20-29]. Finish §2.2 by deriving some more standard properties of T_n. Then prove Corollary 7 to show that each ideal of T_n is finite and complete (Corollary 8) and generalize this results to finite generated T_n-modules (Corollary 10).

1. Non-archimedean absolute value

Definition 1.1. Let K be a field. A map $|\cdot| : K \to \mathbb{R}_{>0}$ is called a *non-archimedean value* if for any $a, b \in K$, the following conditions are satisfied:

(1) $|a| = 0$ if and only if $a = 0$;
(2) $|ab| = |a||b|$;
(3) $|a + b| \leqslant \max\{|a|, |b|\}$.

Correspondingly, there is another map $v : K \to \mathbb{R} \cap \{\infty\}$ called the *valuation* on K,[1] such that for any $a, b \in K$,

(1) $v(a) = \infty$ if and only if $a = 0$;
(2) $v(ab) = v(a) + v(b)$;
(3) $v(a + b) \geqslant \min\{v(a), v(b)\}$.

A non-archimedean value $|\cdot|$ is called *trivial* if $|x| = 1$ for all $x \neq 0$; also, it is called *discrete* if $|K^*|$ is discrete in $\mathbb{R}$.

Due to the strong triangle inequality (3) above, we obtain some unusual properties on the topology of K.

Proposition 1.2. *If $a, b \in K$ such that $|a| \neq |b|$ then*

$$|a + b| = \max\{|a|, |b|\}.$$

Proof. May assume $|a| > |b|$. Then $|a + b| \leqslant \max\{|a|, |b|\} = |a|$ by the strong triangle inequality. On the other hand, $|a| = |(a + b) - b| \leqslant \max\{|a + b|, |b|\} = |a + b|$. This proves the equality. □

The following proposition shows that the property of Cauchy sequence in non-archimedean fields is slightly stronger than that in archimedean fields.

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[1]One can intuitively take $v(a) = -\log|a|$ for example.

Proposition 1.3. *Suppose $a_\nu \in K$. The sequence $\{\sum_{\nu=0}^N a_\nu\}_{N=0}^\infty$ is a Cauchy sequence if and only if* $\lim_{\nu\to\infty}|a_\nu| = 0$.

The preceding considerations show another peculiarity of the topology of K.

Proposition 1.4. *Fix $a \in K$ and $r \in \mathbb{R}_{>0}$. Then all of the following regions*

$$\begin{aligned} D^-(a,r) &= \{x \in K \mid |x-a| < r\}, \\ D^+(a,r) &= \{x \in K \mid |x-a| \leqslant r\}, \\ \partial D(a,r) &= \{x \in K \mid |x-a| = r\}, \end{aligned}$$

are simultaneously open and closed.

In particular, the unit disc $D^+(0,1)$ can be written as a disjoint union $D^-(0,1) \sqcup \partial D(0,1)$ of two open subsets. More generally, we have:

Proposition 1.5. *The topology of K is totally disconnected, i.e. any subset in K consisting of more than just one point is not connected.*

The functions on the underlying topological space K itself infer some information on this weird non-archimedean topology as well.

Example 1.6. Let f be a function on the open (and closed) ball $D^+(a,r)$ defined as

$$f(x) = \begin{cases} 0 & x \in D^-(a,r), \\ 1 & x \in \partial D(a,r). \end{cases}$$

Then f can never be continuous.

The basic principle of rigid analytic geometry is to require that analytic functions on disks admit globally convergent power series expansions. We will discuss the details of the precise definition in subsequent sections.

2. Restricted power series

In this section we set $(K, |\cdot|)$ a complete non-archimedean field with a non-trivial valuation. Denote $\overline{K}$ the algebraic closure of K. It turns out that if L/K is an algebraic extension, then there is a unique way to extend $|\cdot|$ on K to $(L, |\cdot|')$. In fact, one can take for any $\alpha \in L$ that

$$|\alpha|' = |\operatorname{Nm}_{K(\alpha)/K}(\alpha)|^{1/d}, \quad d = \deg \alpha.$$

Given this construction, we observe that if $f(x) = x^n + \cdots + a_0$ is the minimal polynomial of α over K, then $|\alpha|' = |a_0|^{1/n}$. In fact, if L/K is a finite extension, then $(L, |\cdot|)$ is automatically complete. Conversely, for example, $\overline{\mathbb{Q}}_p$ is not complete as an infinite extension of $\mathbb{Q}_p$. So we usually consider the topological ring $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$ instead. Denote

$$\mathbb{B}^n(\overline{K}) := \{(x_1, \ldots, x_n) \in \overline{K}^n \mid |x_i| \leqslant 1\}$$

the closed unit ball.

Lemma 2.1. *Consider the formal power series*

$$f = \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu = \sum_{\nu \in \mathbb{N}^n} c_{\nu_1 \dots \nu_n} \zeta_1^{\nu_1} \dots \zeta_n^{\nu_n} \in K[\![\zeta_1, \dots, \zeta_n]\!].$$

Then f converges on $\mathbb{B}^n(\overline{K})$ if and only if $\lim_{\nu \to \infty} |c_\nu| = 0$.

Definition 2.2. Define the *Tate algebra*

$$\begin{aligned} T_n = K\langle \zeta_1, \dots, \zeta_n \rangle &:= \{ f \in K[\![\zeta_1, \dots, \zeta_n]\!] \mid f \text{ converges on } \mathbb{B}^n(\overline{K}) \} \\ &= \{ f = \sum c_\nu \zeta^\nu \in K[\![\zeta_1, \dots, \zeta_n]\!] \mid |c_\nu| \to 0 \text{ as } \nu \to \infty \}. \end{aligned}$$

Also define the *Gauss norm* $|\cdot|$ on T_n by

$$|f| := \max |c_\nu|, \quad f = \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu \in T_n.$$

The following are some basic properties of the Gauss norm.

(1) $|f| = 0$ if and only if $f = 0$;
(2) $|cf| = |c| \cdot |f|$ for each $c \in K$;
(3) $|fg| = |f| \cdot |g|$;
(4) $|f + g| \leqslant \max\{|f|, |g|\}$.

Here the properties (1)(2)(4) are obviously given by the non-archimedean nature of Tate algebra. And (3) is an algebraic property. To prove (3), it suffices to check the equality for $f_0 = f/|f|$ and $g_0 = g/|g|$ with $|f_0| = |g_0| = 1$. Let us introduce a valuation ring on K, say

$$R = \{ x \in K \mid |x| \leqslant 1 \}, \quad \mathfrak{m} = \{ x \in K \mid |x| < 1 \}, \quad k = R/\mathfrak{m}.$$

There is a natural algebra homomorphism

$$\pi : R\langle \zeta_1, \dots, \zeta_n \rangle \longrightarrow k[\zeta_1, \dots, \zeta_n], \quad f = \sum c_\nu \zeta^\nu \longmapsto \sum \widetilde{c_\nu} \zeta^\nu = \widetilde{f}.$$

Here as $\nu \to \infty$, $|c_\nu| \to 0$ and we take $\widetilde{c_\nu} = 0$ correspondingly. Given $|f_0| = |g_0| = 1$ we infer that $\pi(f_0) \neq 0$ and $\pi(g_0) \neq 0$. Hence $\pi(f_0 g_0) \neq 0$, which implies by definition $|f_0 g_0| = 1$.

Proposition 2.3. *The normed Tate algebra $(T_n, |\cdot|)$ is complete.*

Proof. Given $\sum_{i=0}^\infty f_i$ with $f_i = \sum c_{i\nu} \zeta^\nu \in T_n$ such that $\lim_{i \to \infty} |f_i| = 0$, we have for any ν that $\lim_{i \to \infty} |c_{i\nu}| = 0$. Take

$$c_\nu = \sum_{i=0}^\infty c_{i\nu}, \quad |c_\nu| \leqslant \max\{|c_{i\nu}|\},$$

so $\lim_{\nu \to \infty} |c_\nu| = 0$. The limit point is defined by $f = \sum c_\nu \zeta^\nu = \sum_{i=0}^\infty f_i \in T_n$. □

Corollary 2.4. *Take $f \in T_n$ such that $|f| = 1$. Then f is a unit in $R\langle \zeta_1, \dots, \zeta_n \rangle$ if and only if $\widetilde{f}$ is a unit in $k[\zeta_1, \dots, \zeta_n]$.*

Proof. Following the algebra homomorphism π as above, we see ($\Rightarrow$) is apparent. As for ($\Leftarrow$), if $f = \sum c_\nu \zeta^\nu$ and $\widetilde{f}$ is a unit, then $|f(0)| = |c_0| = 1$ and $|c_\nu| < 1$ for $\nu \neq 0$. Without loss of generality we can take $c_0 = 1$ with $g = \sum_{\nu \neq 0} c_\nu \zeta^\nu$ and $|g| < 1$. Take

$$h^{-1} := f = 1 + g = (1 - g + g^2 - \cdots)^{-1},$$

and then $fh = 1$. This implies that f is a unit. □

Theorem 2.5 (Maximum principle). *For $f \in T_n$, we have*

$$|f(x)| \leqslant |f|, \quad \forall x \in \mathbb{B}^n(\overline{K}),$$

and there exists a point $x_0 \in \mathbb{B}^n(\overline{K})$ such that $|f(x_0)| = |f|$.

Proof. May assume $|f| = 1$ and $f = \sum c_\nu \zeta^\nu$. Then

$$|f(x)| = |\sum c_\nu x^\nu| \leqslant \max\{|c_\nu|\} = |f| = 1.$$

Again, we consider the algebra homomorphism

$$\pi : R\langle \zeta_1, \dots, \zeta_n \rangle \longrightarrow k[\zeta_1, \dots, \zeta_n], \quad \widetilde{f} = \pi(f).$$

So $|f| = 1$ implies $|\widetilde{f}| \neq 0$. There exists $\widetilde{x} \in \overline{k}^n$ with $k = R/\mathfrak{m}$ such that $\widetilde{f}(\widetilde{x}) \neq 0$. Define

$$\overline{R} = \{x \in \overline{K} \mid |x| \leqslant 1\}, \quad \overline{\mathfrak{m}} = \{x \in \overline{K} \mid |x| < 1\}, \quad \overline{k} = \overline{R}/\overline{\mathfrak{m}}.$$

Then one can find a lifting $x \in \mathbb{B}^n(\overline{K})$ of $\widetilde{x}$. Consider the commutative diagram

$$\begin{array}{ccc} R\langle \zeta_1, \dots, \zeta_n \rangle & \xrightarrow{\pi} & k[\zeta_1, \dots, \zeta_n] \\ \downarrow & & \downarrow \\ \overline{R} & \xrightarrow{\pi_1} & \overline{k} \end{array}$$

where the first vertical map is evaluation at x and the second one evaluation at $\widetilde{x}$. Since $\widetilde{f}(\widetilde{x}) \neq 0$ we see $\pi_1(f(x)) \neq 0$. However, $f(x) \notin \overline{\mathfrak{m}}$, which deduces $|f(x)| = 1$. This completes the proof. □

The Tate algebra Tn has many properties in common with the polynomial ring in n variables over K, as we will see. The key tool for proving all these properties is Weierstrass theory, which we will explain now and which is quite analogous to Weierstrass theory in the classical complex case. In particular, we will establish Weierstrass division, a division process similar to Euclid's division on polynomial rings. In Weierstrass theory the role of monic polynomials is taken over by so-called distinguished restricted power series, or later by so-called Weierstrass polynomials.

Definition 2.6. A restricted power series $g = \sum_{\nu=0}^{\infty} g_\nu \zeta_n^\nu \in T_n$ with coefficients $g_\nu \in T_{n-1}$ is called ζ_n*-distinguished* of some order $s \in \mathbb{N}$ if the following hold:

(1) g_s is a unit in T_{n-1};
(2) $|g_s| = |g|$ and $|g_s| > |g_v|$ for $v > s$.

The following lemma dictates that there exists an algorithm to convert non-ζ_n-distinguished elements into ζ_n-distinguished elements.

Lemma 2.7. *Given $f_1, \dots, f_r \in T_n$, there is a continuous automorphism*

$$\sigma : T_n \longrightarrow T_n$$

such that $\sigma(f_i)$ is ζ_n-distinguished for each i. Furthermore, for all $f \in T_n$,

$$|\sigma(f)| = |f|.$$

Theorem 2.8 (Weierstrass division)**.** *Let $g \in T_n$ be a ζ_n-distinguished element of order s. Then for any $f \in T_n$ there exists a unique $q \in T_n$ and $r \in T_{n-1}[\zeta_n]$ such that*

$$f = qg + r, \quad \deg r < s.$$

Furthermore,

$$|f| = \max\{|qg|, |r|\}.$$

Proof. Without loss of generality we assume $|g| = 1$. If $|f| < \max\{|qg|, |r|\}$ then $|qg| = |r| > |f|$. Assume that $1 = |qg| = |r| > |f|$. Consider

$$\pi : R\langle\zeta_1, \dots, \zeta_n\rangle \longrightarrow k[\zeta_1, \dots, \zeta_n].$$

Thus, $\pi(qg + r) = \widetilde{q}\widetilde{g} + \widetilde{r} = \widetilde{f} = 0$ but $\widetilde{q}, \widetilde{g}, \widetilde{r} \neq 0$. This leads to a contradiction. So $|f| = \max\{|qg|, |r|\}$. Moreover, if $f = q_1 g + r_1 = q_2 g + r_1$ then $(q_1 - q_2)g + (r_1 - r_2) = 0$, which implies $q_1 = q_2$ and $r_1 = r_2$. This proves the uniqueness.

For the existence, may assume $|f| = 1$ and denote

$$\varepsilon = |g - \sum_{i=0}^{s} g_i \zeta_n^i| < 1.$$

We claim that there are $q_1, f_1 \in T_n$ and $r \in T_{n-1}[\zeta_n]$ such that

$$f = qg + r + f_1, \quad \deg r < s, \quad |q|, |r| \leqslant |f|, \ |f_1| \leqslant \varepsilon |f|.$$

- If $f \in T_{n-1}[\zeta_n]$, let $g' = \sum_{i=0}^{s} g_i \zeta_n^i$ with $|g'| = 1$. By the Euclidean division in $T_{n-1}[\zeta_n]$ we have $f = qg' + r$ with $\deg r < s$ and $|f| = \max\{|qg'|, |r|\}$. Therefore,

$$f = qg + r + (qg' - qg), \quad |qg' - qg| = |q| \cdot |g' - g| \leqslant \varepsilon |f|.$$

- If $f \notin T_{n-1}[\zeta_n]$, then

$$f = \lim_{m \to \infty} f_m, \quad f_m \in T_{n-1}[\zeta_n].$$

  From the previous point we see $f_m = q_m g' + r_m$ for each m. And

$$|f_{m+1} - f_m| = |(q_{m+1} - q_m)g' + (r_{m+1} - r_m)| \to 0$$

  as $m \to \infty$. We infer that $|q_{m+1} - q_m| \to 0$ and $|r_{m+1} - r_m| \to 0$. One can take

$$q = \lim_{m \to \infty} q_m, \quad r = \lim_{m \to \infty} r_m.$$

  Then $f = qg' + r$.

Now we apply the induction. Let $f_0 = f$ and suppose $f_0 = q_0 g + r_0 + f_1$, where $|q_0|, |r_0| \leqslant 1$ and $|f_1| \leqslant \varepsilon < 1$. Similarly, we have $f_1 = q_1 g + r_1 + f_2$ where $|q_1|, |r_1| \leqslant \varepsilon$ and $|f_2| \leqslant \varepsilon^2$. Using this process, we get

$$f = g \cdot \sum_{i=0}^{\infty} q_i + \sum_{i=0}^{\infty} r_i.$$

This completes the proof of Weierstrass division. □

Corollary 2.9 (Weierstrass preparation theorem)**.** *Let $g \in T_n$ be ζ_n-distinguished of order s. Then there is a unique monic $w \in T_{n-1}[\zeta_n]$ with $\deg w = s$, such that $g = e \cdot w$, where e is a unit in T_n. Furthermore, $|w| = 1$ and w is ζ_n-distinguished.*

Corollary 2.10. *$T_1 = K\langle\zeta_1\rangle$ is a Euclidean domain.*

Corollary 2.11 (Noether normalization)**.** *For any proper ideal* $\mathfrak{a} \subsetneq T_n$*, there is a* K*-algebra monomorphism* $T_d \hookrightarrow T_n$ *for some* $d \in \mathbb{N}$ *such that* $T_d \hookrightarrow T_n \twoheadrightarrow T_n/\mathfrak{a}$ *is a finite monomorphism.*

Proof. Choose $g \in \mathfrak{a}\backslash\{0\}$. Applying a suitable automorphism to T_n, we can assume g is ζ_n-distinguished of order s by Lemma 2.7. Consider the natural homomorphism

$$T_{n-1} \hookrightarrow T_n \to T_n/(g).$$

For $\zeta_n \in T_n$, by Weierstrass division $\zeta_n = qg + r$ where $\deg r < 1$. Hence $r \in T_{n-1}$. Also, $\widetilde{\zeta_n} = \widetilde{r}$, so $\widetilde{\zeta_n}$ is integral over T_{n-1}. Hence $T_{n-1} \to T_n/(g)$ is finite.

Now consider the post-composite

$$f : T_{n-1} \to T_n/(g) \to T_n/\mathfrak{a},$$

with $\mathfrak{a}_1 := \ker f$. If $\mathfrak{a}_1 = 0$ we are done. Otherwise choose $g_1 \in \mathfrak{a}$ and repeat the operation to $\mathfrak{a}_1 \subseteq T_{n-1}$. After finitely many iterations there is a finite monomorphism $T_d \to T_n/\mathfrak{a}$ for some d. □

Corollary 2.12. *Suppose* $\mathfrak{m} \subseteq T_n$ *is a maximal ideal. Then* $T_n/\mathfrak{m}$ *is finite field extension over* K*.*

Proof. Via the previous corollary $T_d \hookrightarrow T_n/\mathfrak{m}$ is a finite monomorphism. Note that the target $T_n/\mathfrak{m}$ is a field, which implies that T_d is a field as well. This forces d to be 0, i.e. $T_d = T_0 = K$. □

Corollary 2.13. *The map*

$$\mathbb{B}^n(\overline{K}) \longrightarrow \operatorname{Spm} T_n, \quad x \longmapsto \mathfrak{m}_x = \{f \in T_n \mid f(x) = 0\}$$

is surjective. Here $\operatorname{Spm} T_n$ *denotes the set of maximal ideals in* T_n*.*

Proof. If $x = (x_1, \ldots, x_n) \in \mathbb{B}^n(\overline{K})$, then the evaluation map

$$T_n \longrightarrow \overline{K}, \quad x \longmapsto f(x)$$

induces an isomorphism $T_n/\mathfrak{m}_x \simeq K(x_1, \ldots, x_n)$. The right hand side is a field and hence $\mathfrak{m}_x$ is a maximal ideal. We are to prove the surjectivity. For any $\mathfrak{m} \in \operatorname{Spm} T_n$,

$$\varphi : T_n \twoheadrightarrow T_n/\mathfrak{m} \hookrightarrow \overline{K}$$

is a finite homomorphism. Let $x_i := \varphi(\zeta_i) \in \overline{K}$. It turns out that $\mathfrak{m} = \mathfrak{m}_{(x_1,\ldots,x_n)}$ and then the surjectivity follows. □

Proposition 2.14. *The Tate algebra* T_n *is noetherian.*

Proof. We prove by using the induction. Note that $T_0 = K$ is noetherian. Assume now T_{n-1} is noetherian. For each $\mathfrak{a} \subsetneq T_n$ proper ideal, we can choose $g \in \mathfrak{a}$ and assume g is ζ_n-distinguished by Lemma 2.7. Consider the integral homomorphism $T_{n-1} \to T_n/(g)$, and we deduce that $T_n/(g)$ is a noetherian T_{n-1}-module. Consequently, $\mathfrak{a}/(g)$ is a noetherian module as well. It follows that $\mathfrak{a}/(g)$ is a finitely generated T_{n-1}-module, so that $\mathfrak{a}$ is finitely generated on T_n. Hence T_n is noetherian. □

Proposition 2.15. *The Tate algebra* T_n *is a factorial integral domain (and hence normal, i.e. integrally closed in its fractional field).*

Proof. Again we apply the induction. The case where $T_0 = K$ is clear. Assume T_{n-1} is factorial and so also is $T_{n-1}[\zeta_n]$. For each $f \in T_n$, f is ζ_n-distinguished. By Weierstrass preparation theorem (Corollary 2.9) $f = e \cdot w$ for some $w \in T_{n-1}[\zeta_n]$. Moreover,

$$f = e \cdot (w_1 \cdots w_r),$$

where each $w_i \in T_{n-1}[\zeta_n]$ is a prime element. It suffices to show that each w_i is prime in T_n. This is clear since there is a natural isomorphism

$$T_{n-1}[\zeta_n]/(w) \xrightarrow{\sim} T_n/(w).$$

The proof for normality is very similar to that $\mathbb{Z}$ is normal in $\mathbb{Q}$, so we choose to omit it. □

Proposition 2.16. *The Tate algebra T_n is Jacobson, i.e. for any $\mathfrak{a} \subsetneq T_n$,*

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{m} \in \operatorname{Max} T_n \\ \mathfrak{a} \subseteq \mathfrak{m}}} \mathfrak{m}.$$[2]

Proof. We can reduce the problem to the case that $\mathfrak{a}$ is a prime ideal. Whenever $\mathfrak{a} = 0$, let $f \in \bigcap_{\mathfrak{m} \in \operatorname{Max} T_n} \mathfrak{m}$. Then $f(x) = 0$ for each $x \in \mathbb{B}^n(\overline{K})$. It follows that $f = 0$. Assume $\mathfrak{a} \neq 0$. Consider the integral monomorphism $T_d \hookrightarrow T_n/\mathfrak{a}$ given by Noether normalization (Corollary 2.11). For each maximal ideal $\mathfrak{m} \subseteq T_d$, there exists a maximal ideal $\mathfrak{m}' \subseteq T_n/\mathfrak{a}$ such that $\mathfrak{m} = \mathfrak{m}' \cap T_d$. If $\mathfrak{q} = \bigcap_{\mathfrak{m} \in \operatorname{Max} T_n/\mathfrak{a}} \mathfrak{m} \subseteq T_n/\mathfrak{a}$, then

$$\mathfrak{q} \cap T_d = \bigcap_{\mathfrak{m} \in \operatorname{Max} T_n/\mathfrak{a}} (\mathfrak{m} \cap T_d) = \bigcap_{\mathfrak{m}' \in \operatorname{Max} T_d} \mathfrak{m}' = 0.$$

Therefore, $\mathfrak{q} \cap T_d = 0$. If $\mathfrak{q} \neq 0$, there exists $0 \neq f \in \mathfrak{q}$ integral over T_d, with $f^r + a_1 f^{r-1} + \cdots + a_r = 0$ with $a_i \in T_d$ and $a_r \neq 0$. But

$$T_d \ni a_r = -f^r - a_1 f^{r-1} - \cdots - a_{r-1} f \in \mathfrak{q}, \quad T_d \cap \mathfrak{q} = 0,$$

which leads to $a_r = 0$, a contradiction. Hence $f = 0$ and $q = 0$. □

Proposition 2.17. *For each maximal ideal $\mathfrak{m} \subseteq T_n$,*

$$\operatorname{ht}(\mathfrak{m}) = n.$$

And $\mathfrak{m}$ is generated by n elements. In particular,

$$\operatorname{Krull}\dim T_n = n.$$

3. Ideals in Tate algebras

The most important feature of ideals in $T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle$ is that all ideals in T_n are closed with respect to the non-archimedean topology of K. In particular, all ideals in T_n are completed.

Definition 3.1. Let R be a ring. A *ring norm* on R is a map $|\cdot| : R \to \mathbb{R}_{>0}$ such that

(1) $|a| = 0$ if and only if $a = 0$;
(2) $|ab| \leqslant |a| \cdot |b|$;
(3) $|a + b| \leqslant \max\{|a|, |b|\}$;
(4) $|1| \leqslant 1$.

[2]Recall that $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals containing $\mathfrak{a}$.

The ring norm is called *multiplicative* if

(2') $|ab| = |a| \cdot |b|$.

Note that in particular, a multiplicative norm is such that $|1| = 1$.

Definition 3.2. Let R be a ring equipped with a multiplicative ring norm $|\cdot|$, such that $|a| \leqslant 1$ for each $a \in R$.

(1) R is called a *B-ring* if

$$\{a \in R \mid |a| = 1\} \subseteq R^*.$$

Namely, all norm one elements of R are invertible.

(2) R is called *bald* if

$$\sup\{|a| : a \in R, |a| < 1\} < 1.$$

We remark on the bald property that any element of R with norm 1 cannot be approximated by a family of elements in R with norm < 1.

Proposition 3.3. *Let K be a field and R a valuation ring. Take a sequence $\{a_0, a_1, \ldots\}$ in R such that $\lim_{n\to\infty} a_n = 0$. Then the smallest subring containing $\{a_n\}_{n=0}^{\infty}$ is bald.*

Proof. Consider the smallest subring of R, denoted by S. Then

$$S = \begin{cases} \mathbb{F}_p & \operatorname{char}(R) = p > 0, \\ \mathbb{Z} & \operatorname{char}(R) = 0. \end{cases}$$

Let $R' = S[a_0, a_1, \ldots]$. We first observe that S is bald: if $S = \mathbb{F}_p$ then all valuations on finite fields are trivial; if $S = \mathbb{Z}$ then $\{a \in \mathbb{Z} : |a| < 1\}$ is a principal ideal in $\mathbb{Z}$, whose generator has norm < 1. Also, suppose there exists $\varepsilon \in \mathbb{R}$ such that $|a_n| \leqslant \varepsilon < 1$ for all $n \in \mathbb{N}$. Then $R' = S[a_0, a_1, \ldots]$ is bald. Since $\{a_n\}_{n=0}^{\infty}$ is a zero sequence, only finitely many $\{a_i\}_{i\in I}$ satisfy $|a_i| = 1$. Alternatively, there is some $\varepsilon \in \mathbb{R}$ such that $|a_n| \leqslant \varepsilon < 1$ for $n \notin I$. It boils down to show that if $S_1 \subseteq R$ is bald, then for any $a \in R$ with $|a| = 1$, $S[a]$ is bald as well. Take

$$\varepsilon := \sup\{|a| : a \in S_1, |a| < 1\} < 1.$$

Without loss of generality assume S_1 is a B-ring, i.e. $\{a \in S_1 \mid |a| = 1\} \subseteq S_1^*$. (If not, can localize S_1 towards the multiplicative subset $\{a \in S_1 \mid |a| = 1\}$ to get a B-ring.) Then for $\mathfrak{m}' = \{x \in S_1 \mid |x| < 1\}$, we have a local ring $(S_1, \mathfrak{m}')$. Also, $(R, \mathfrak{m})$ is a local ring. So $\mathfrak{m}' = \mathfrak{m} \cap S$. As there is a monomorphism $S_1 \hookrightarrow R$, we have $\widetilde{S} := S_1/\mathfrak{m}' \hookrightarrow R/\mathfrak{m} = k$. Then for $a \in R$ with $|a| = 1$, we get $0 \neq \widetilde{a} \in k$. And $\widetilde{a}$ is either transcendental or algebraic over $\widetilde{S}$.

- Suppose $\widetilde{a}$ is transcendental over $\widetilde{S}$. For each $p = \sum_{i=0}^{r} c_i a^i \in S_1[a]$ with $c_i \in S$, if $|p| < 1$ then

$$\widetilde{p} = \sum_{i=0}^{r} \widetilde{c_i}\widetilde{a}^i = 0, \quad \widetilde{c_i} \in \widetilde{S}.$$

This deduces $\widetilde{c_i} = 0$, and hence $|c_i| < 1$. Thus,

$$|p| \leqslant \max\{|c_i|\} \leqslant \varepsilon < 1.$$

- Suppose $\widetilde{a}$ is algebraic over $\widetilde{S}$. Choose

$$g(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in S_1[T],$$

such that $\widetilde{g}$ is the minimal polynomial of $\widetilde{a}$ over $\widetilde{S}$. Then $\widetilde{g}(\widetilde{a}) = 0$ implies $|g(a)| < 1$. Let

$$\varepsilon_0 := \max\{|g(a)|, \varepsilon\} < 1.$$

For each $f \in S_1[T]$ with $|f(a)| < 1$, we need to show that $|f(a)| \leqslant \varepsilon_0$ (and therefore $S_1[a]$ is a bald ring). By Euclid division, $f = qg + r$ with $q, r \in S_1[T]$ and $\deg r < \deg g$. In particular,

$$f(a) = q(a) \cdot g(a) + r(a),$$

where

$$|f(a)| < 1, \quad |q(a)| \leqslant 1, \quad |g(a)| \leqslant \varepsilon_0.$$

If $|f(a)| > \varepsilon_0$, then the only possibility is read as $|f(a)| = |r(a)|$. If $|r| < 1$ and $r = \sum_{i=0}^{\infty} b_i x^i$ then $|r(a)| = |\sum_{i=0}^{\infty} b_i a^i| \leqslant \max\{|b_i|\}$. If $|r| = 1$ then $\widetilde{r} \neq 0$. But $\widetilde{r}(\widetilde{a}) = 0$, which implies a contradiction. Hence $\widetilde{g} \mid \widetilde{r}$.

This completes the proof. □

We summarize from the proof above that given a bald subring $R' \subseteq R$, by localizing R' with respect to the multiplicative subset $\{x \in R' \mid |x| = 1\}$, we get a B-ring R'' such that

$$R' \subseteq R'' \subseteq R.$$

Moreover, R'' is bald as R' is bald. If R is further complete, and if R'' is a bald B-ring, then so also is $\widehat{R''}$. We check the B-ring property. For each limit point $x = (x_1, x_2, \ldots) \in \widehat{R''}$ with $|x| = 1$, we have $x_i \in R''$. Assume $|x_i| = 1$ for all i. Then one can take $x^{-1} = (x_1^{-1}, x_2^{-1}, \ldots)$ as R'' is a B-ring. The baldness follows a similar argument.

Definition 3.4. Let V be a K-vector space. A *norm* on V is a map $|\cdot| : V \to \mathbb{R}_{>0}$ such that

(1) $|x| = 0$ if and only if $x = 0$;
(2) $|x + y| \leqslant \max\{|x|, |y|\}$;
(3) $|cx| = |c| \cdot |x|$ for $c \in K$.

Let V be a complete normed K-vector space. A system $\{x_\nu\}_{\nu \in N}$ of elements in V, where the index set N is either finite or countably infinite, is called a *topological orthonormal basis* of V, if

(1) $|x_\nu| = 1$ for all $\nu \in N$;
(2) each $x \in V$ can be written as a convergent power series $x = \sum_{\nu \in N} c_\nu x_\nu$ with $c_\nu \in K$;
(3) for $x = \sum_{\nu \in N} c_\nu x_\nu$, we always have $|x| = \max_{\nu \in N} |c_\nu|$.

Example 3.5. We have a natural topological orthonormal basis $\{\zeta^\nu\}_{\nu \in \mathbb{N}^n} \subseteq T_n$, as each $f \in T_n$ can be written as $f = \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu$.

We introduce two more notations for a K-vector space V that

$$V^\circ := \{x \in V \mid |x| \leqslant 1\},$$

$$\widetilde{V} := V^\circ / \{x \in V \mid |x| < 1\}.$$

Theorem 3.6. *Let K be a complete non-archimedean field with the residue field k, V a complete normed K-vector space, and R a valuation ring on K. Let $(x_\nu)_{\nu \in N}$ be a topological orthonormal basis on V. Take*

$$y_\mu = \sum_{\nu \in N} c_{\mu\nu} x_\nu \in V^\circ, \quad \mu \in M.$$

where the smallest subring of R containing $\{c_{\mu\nu}\}$ is bald. Then, if the residue classes $\widetilde{y_\mu} \in \widetilde{V}$ form a k-basis of $\widetilde{V}$, the elements $\{y_\mu\}_{\mu \in M}$ form an orthonormal basis of V as well.

Proof. The systems $(\widetilde{x}_v)_{v \in N}$ and $(\widetilde{y}_\mu)_{\mu \in M}$ form a k-basis of $\widetilde{V}$. So M and N have the same cardinality, and M is at most countable. In particular, $(y_\mu)_{\mu \in M}$ is an orthonormal basis of a subspace $V' \subset V$. Now let S be the smallest complete B-ring in R containing all coefficients $c_{\mu\nu}$. Then S is bald by our assumption; let $\varepsilon = \sup\{|a| : a \in S, |a| < 1\}$. Setting

$$V'_S = \widehat{\sum_{\mu \in M}} S y_\mu, \quad V_S = \widehat{\sum_{\nu \in N}} S x_\nu,$$

where $\widehat{\sum}$ means the completion of the usual sum, we have $V'_S \subseteq V_S$, and we claim that, in fact, $V'_S = V_S$. To verify this, let us first look at reductions. If $\mathfrak{m} \subseteq S$ denotes the unique maximal ideal, we set

$$\widetilde{S} = S/\mathfrak{m}, \quad V'_{\widetilde{S}} = V'_S/\mathfrak{m}V'_S, \quad V_{\widetilde{S}} = V_S/\mathfrak{m}V_S.$$

Then $\widetilde{S}$ is a subfield of the residue field k of R, and we have

$$\widetilde{V}' = V'_{\widetilde{S}} \otimes_{\widetilde{S}} k, \quad \widetilde{V} = V_{\widetilde{S}} \otimes_{\widetilde{S}} k.$$

From $\widetilde{V}' = \widetilde{V}$ and $V'_{\widetilde{S}} \subseteq V_{\widetilde{S}}$ we get $V'_{\widetilde{S}} = V_{\widetilde{S}}$ as fields extensions are faithfully flat. The latter implies that, for any x_ν, there is an element $z_v \in V'_S$ satisfying $|x_v - z_v| \leqslant \varepsilon$. Then, more generally, for any $x \in V_S$, there is an element $z \in V'_S$ with $|z| = |x|$ and $|x - z| \leqslant \varepsilon |x|$. But then, as V'_S and V_S are complete, we get $V'_S = V_S$ by iteration. □

Now we want to apply Theorem 3.6 to Tate algebras with $V = T_n$.

Corollary 3.7. *Let $\mathfrak{a} \lhd T_n$ be an ideal. Then there are generators $a_1, \ldots, a_r$ of $\mathfrak{a}$ such that*

(i) *for each i, $|a_i| = 1$, and*
(ii) *for each $f \in \mathfrak{a}$, there are $f_1, \ldots, f_r \in T_n$ such that $f = \sum_{i=1}^r f_i a_i$ with $|f_i| \leqslant |f|$.*

Proof. Let $\widetilde{\mathfrak{a}}$ be the reduction of $\mathfrak{a}$, i.e. the image of $\mathfrak{a} \cap R\langle\zeta\rangle$ under the reduction map $R\langle\zeta\rangle \to k[\zeta]$ where R is the valuation ring of K. Then $\widetilde{\mathfrak{a}}$ is an ideal in the Noetherian ring $k[\zeta]$ and, hence, finitely generated, say by the residue classes $\widetilde{a}_1, \ldots, \widetilde{a}_r$ of some elements $a_1, \ldots, a_r \in \mathfrak{a}$ having norm equal to 1. As the elements $\zeta^v \widetilde{a}_i$ for $\nu \in \mathbb{N}^n$, $i = 1, \ldots, r$, generate $\widetilde{\mathfrak{a}}$ as a k-vector space, we can find a system $(y_\mu)_{\mu \in M'}$ of elements of type $\zeta^v a_i \in \mathfrak{a}$ such that its residue classes form a k-basis of $\mathfrak{a}$. Adding monomials of type ζ^v for all $v \in \mathbb{N}^n$, we can enlarge the system to a system $(y_\mu)_{\mu \in M}$ such that its residue classes form a k-basis of $k[\zeta]$.

On the other hand, let us consider the system $(\zeta^\nu)_{\nu \in \mathbb{N}^n}$ of all monomials in T_n; it is an orthonormal basis of T_n and its reduction forms a k-basis of $k[\zeta]$. Now apply Proposition 3.3 and Theorem 3.6. To write the elements y_μ as (converging) linear combinations of the

ζ^ν, we need only the coefficients of the series $a_1, \ldots, a_r$. As these form a zero sequence, we see that $(y_\mu)_{\mu \in M}$ is an orthonormal basis, the same being true for $(\zeta^\nu)_{\nu \in \mathbb{N}^n}$.

We want to show that the elements $a_1, \ldots, a_r$ have the required properties. Choose $f \in \mathfrak{a}$. Then, since $(y_\mu)_{\mu \in M}$ is an orthonormal basis of T_n, there is an equation $f = \sum_{\mu \in M} c_\mu y_\mu$ with certain coefficients $c_\mu \in K$ satisfying $|c_\mu| \leqslant |f|$. Writing $f' = \sum_{\mu \in M'} c_\mu y_\mu$, the choice of the elements $y_\mu, \mu \in M'$, implies that we can write $f' = \sum_{i=1}^r f_i a_i$ with certain elements $f_i \in T_n$ satisfying $|f_i| \leqslant |f|$. In particular, $f' \in \mathfrak{a}$, and we are done if we can show $f = f'$. To justify the latter equality, we may replace f by

$$f - f' = \sum_{\mu \in M - M'} c_\mu y_\mu \in \mathfrak{a}$$

and thereby assume $c_\mu = 0$ for $\mu \in M'$. Then, if $f \neq 0$, there is an index $\mu \in M - M'$ with $c_\mu \neq 0$. Assuming $|f| = 1$, we would get a non-trivial equation

$$\widetilde{f} = \sum_{\mu \in M - M'} \widetilde{c}_\mu \widetilde{y}_\mu$$

for the element $\widetilde{f} \in \widetilde{\mathfrak{a}}$, which however, contradicts the construction of the elements $\widetilde{y}_\mu$. □

This immediately implies the result we have mentioned at the beginning of this section.

Corollary 3.8. *For any ideal $\mathfrak{a} \lhd T_n$, $\mathfrak{a}$ is complete, and hence closed in T_n.*

Proof. Choose $a_1, \ldots, a_r$ as in the proof above. If $f = \sum_{\lambda=0}^\infty f_\lambda \in T_n$ for $f_k \in \mathfrak{a}$, then

$$f_\lambda = \sum_{i=0}^r f_{\lambda i} a_i, \quad |f_{\lambda i}| \leqslant |f_\lambda|.$$

Hence

$$f = \sum_{i=1}^r \sum_{\lambda=0}^\infty f_{\lambda i} a_i,$$

and in particular $f \in \mathfrak{a}$. This shows the completeness. □

Corollary 3.9. *For each $\mathfrak{a} \lhd T_n$, $\mathfrak{a}$ is strictly closed, i.e. for each $f \in T_n$ there exists $a_0 \in \mathfrak{a}$ such that*

$$|f - a_0| = \inf_{a \in \mathfrak{a}} |f - a|.$$

Proof. Resume the notations before. Choose $(y_\mu)_{\mu \in M}$ as above. Take

$$f = \sum_{\mu \in M} c_\mu y_\mu, \quad a_0 = \sum_{\mu \in M'} c_\mu y_\mu.$$

For any $a = \sum_{\mu \in M'} d_\mu y_\mu \in \mathfrak{a}$ say, and then

$$|f - a| = \left| \sum_{\mu \in M - M'} c_\mu d_\mu + \sum_{\mu \in M'} (c_\mu - d_\mu) y_\mu \right| = \max_{\substack{\mu \in M - M' \\ \nu \in M'}} \{|c_\mu|, |c_\nu - d_\nu|\}$$

and

$$|f - a_0| = \max_{\mu \in M - M'} \{|c_\mu|\}.$$

These deduce the desired result $|f - a_0| \leqslant |f - a|$. □

We remark that the strict closedness is an analogue of the compact notion in the non-archimedean topology.

Summary 3.10. Recall that elements in Tate algebra T_n can be regarded as continuous functions from $\mathbb{B}^n(\overline{K})$ to $\overline{K}$. Moreover, it satisfies the following properties.

(1) The maximum principle: for each $f \in T_n$ and any $x \in \mathbb{B}^n(\overline{K})$ we have
$$|f(x)| \leqslant |f| = |\sum c_\nu \zeta^\nu| = \max\{|c_\nu|\}.$$
And there exists x_0 such that $|f(x_0)| = |f|$.

(2) There is a surjective map
$$\varphi : \mathbb{B}^n(\overline{K}) \longrightarrow \operatorname{Max} T_n, \quad x \longmapsto \mathfrak{m}_x = \{f \in T_n \mid f(x) = 0\}.$$

(3) By Weierstrass division theory, we have some algebraic properties: T_n is noetherian, Jacobson, factorial, and normal.

(4) All ideals in T_n are closed.

References

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