Notation: K: complete non-archi. field. $\bar{K}$: alg. closure.
k: residue field

Rmk. Since open disk is clopen, K is totally disconnected.

2.2. Res. power series.

Denote $B^n(\bar{K}) = \mathcal{O}_{\bar{K}}^n$.

Lem. 1. A formal p.s. $f = \sum c_{\nu_1 \cdots \nu_n} \zeta_1^{\nu_1} \cdots \zeta_n^{\nu_n} \in K[\![\zeta_1, \cdots, \zeta_n]\!]$ (denote $\nu = (\nu_1, \cdots, \nu_n)$)

converges on $B^n(\bar{K}) \Leftrightarrow \lim_{|\nu| \to \infty} |c_\nu| \to 0$

P.f. For $\forall x \in B^n(\bar{K})$, $f(x)$ converge. $\Leftrightarrow |c_\nu| \cdot |x|^{|\nu|} \to 0$.

$\therefore f$ conv. at any $x \in B^n(\bar{K}) \Leftrightarrow f$ conv. at $(1, 1, \cdots, 1)$

$\Leftrightarrow |c_\nu| \to 0$.

Denote $T_n = K\langle \zeta_1, \cdots, \zeta_n \rangle := \{ \sum c_\nu \cdot \zeta^\nu \in K[\![\zeta_1, \cdots, \zeta_n]\!] \mid \lim_{|\nu| \to \infty} |c_\nu| = 0 \}$.

$= \{ f \in K[\![\zeta_1, \cdots, \zeta_n]\!] \mid f \text{ conv. on } B^n(\bar{K}) \}$.

$T_0 := K$.

Gauss norm. for $f \in T_n$, $|f| := \max |c_\nu|$. ($f = \sum c_\nu \cdot \zeta_\nu$).

Prop. $|\cdot|$ is a norm. i.e. (1) $|f| = 0 \Leftrightarrow f = 0$.

(2) for any $c \in K$, $|cf| = |c| \cdot |f|$ } trivial.

(3). $|fg| = |f| \cdot |g|$.

(4). $|f+g| \leq \max\{|f|, |g|\}$.

Pf. (3). WLOG Assume $|f| = |g| = 1$. $|fg| \le 1$

Consider $T_n \to k[\xi_1, \cdots, \xi_n]$

$f \to \bar{f}$

$\overline{fg} = \bar{f} \cdot \bar{g} \neq 0$. $\therefore |fg| = 1$.

(4). $f = \sum c_\nu \xi^\nu$ $g = \sum d_\nu \xi^\nu$

$\max_\nu |c_\nu + d_\nu| \le \max_\nu \{ |c_\nu|, |d_\nu| \} = \max\{|f|, |g|\}$.

Prop. 3. T_n is complete w.r.t. $|\cdot|$.

Pf. Suppose $\{f_n\}$ satisfies $|f_n| \to 0$. Need to prove $\sum f_n \in T_n$.

Denote $f_n = \sum c_{n,\nu} \xi^\nu$. $\Leftarrow$ ① $c_\nu := \sum_n c_{n,\nu} \in k$. ② $\lim_{|\nu| \to \infty} |c_\nu| = 0$.

① $|c_{n,\nu}| \le |f_n| \to 0$.

② For any $\varepsilon > 0$, $\exists N$ s.t. when $n > N$, $|f_n| < \varepsilon$.

For $f_1, \cdots, f_N$, $\exists \nu'$ s.t. when $\begin{cases} \nu > \nu' \\ 1 \le i \le N \end{cases}$ $|c_{i,\nu}| < \varepsilon$.

$\therefore$ when $\nu > \nu'$, $|c_\nu| = |\sum c_{n,\nu}| = \left| \sum_{i=1}^{N} c_{i,\nu} + \sum_{i > N} c_{i,\nu} \right| < \varepsilon$.

Cor. 4. $f \in T_n$ with $|f| = 1$ is a unit $\Leftrightarrow \bar{f} \in k^*$. (in $k[\xi_1, \cdots, \xi_n]$)

(In general, f is a unit $\Leftrightarrow |f - f(0)| < |f(0)|$.)

P.f. "$\Rightarrow$" If $f \in T_n^*$, $\exists g \in T_n^*$ s.t. $fg = 1$. $\therefore \bar{f} \cdot \bar{g} = 1$. $\therefore \bar{f} \in k^*$.

"⇐" denote $f(0)=c$. $g=f-c$. $\therefore f=c+g$. $|g|<1$. $\therefore f$ is invertible

$c(1+c^{-1}g)$

Prop. 6. For $f\in T_n$, then for $\forall x\in B^n(\bar{k})$. $|f(x)|\le|f|$. and $\exists x$ s.t. $|f(x)|=|f|$

Pf. $|f(x)|\le|f|$ ✓. For the second assume $|f|=1$.

$\therefore \bar{f}\in k[\zeta_1,\cdots,\zeta_n]$ $\therefore \exists t\in k^n$, s.t. $\bar{f}(t)\neq 0$, pick any lift x of t $(x\in B^n(\bar{k}))$. $|f(x)|=1$. $(\overline{f(x)}=\bar{f}(t)\neq 0)$.

Def. for $g=\sum_{\nu=0}^{\infty} g_\nu\cdot\zeta_n^\nu\in T_n$ with $g_\nu\in T_{n-1}$ is called ζ_n-distinguished if $\exists s$. s.t.

(i) $g_s\in T_{n-1}^*$

(ii) $|g|=|g_s|$, and for $\nu>s$, $|g_s|>|g_\nu|$.

Lem. 7. Given $\forall f_1,\cdots,f_r\in T_n$, $\exists$ cont. aut. (keeps the norm).

$$\sigma: T_n\longrightarrow T_n \qquad \zeta_i\longrightarrow\begin{cases}\zeta_i+\zeta_n^{\alpha_i} & i<n\\ \zeta_n & i=n.\end{cases}$$

with suitable $\alpha_i\in\mathbb{N}$ $(1\le i\le n-1)$ s.t. $\sigma(f_i)$ are dist. $(1\le i\le r)$.

Pf. σ^{-1}: $\zeta_i\longrightarrow\zeta_i-\zeta_n^{\alpha_i}$ $(i<n)$. $\therefore \sigma\in \mathrm{Aut}(T_n)$.

$\therefore |\sigma(f)|\le|f|$. $|\sigma^{-1}(f)|\le|f|$ $(\forall f)$. $\therefore |\sigma(f)|=|f|$ $(\forall f)$ $\therefore \sigma$ cont.

Goal: find α_i $(1\le i\le r)$ s.t. $\sigma(f_i)$ are dist.

Baby case: assume there is only one f. $|f|=1$.

Denote $f=\sum c_\nu\zeta^\nu$. $\bar{f}=\sum_{\nu\in N}\bar{c}_\nu\cdot\zeta^\nu$ (N: nonzero indexes in $\bar{f}$)

Choose t greater than any ν_i occuring in any $\nu \in N$.

Let $\alpha_1 = t^{n-1}, \cdots, \alpha_{n-1} = t$.

$$\overline{\sigma(f)} = \sum_{\nu \in N} \overline{c_\nu} (\xi_1 + \xi_n^{\alpha_1})^{\nu_1} \cdots (\xi_{n-1} + \xi_n^{\alpha_{n-1}})^{\nu_{n-1}} \cdot \xi_n^{\nu_n}.$$

$$= \sum_{\nu \in N} \overline{c_\nu}\, \xi_n^{\alpha_1 \nu_1 + \cdots + \alpha_{n-1}\nu_{n-1} + \nu_n} + g \quad (\deg_{\xi_n} g < \sum_{i<n} \alpha_i \nu_i + \nu_n).$$

Due to the choice $\alpha_1, \cdots, \alpha_{n-1}$, $\sum_{i<n} \alpha_i \nu_i + \nu_n$ are all different.

$\therefore$ their maximum s is achieved at a single $\overline{\nu} \in N$.

$\therefore \overline{\sigma(f)} = \overline{c_{\overline{\nu}}} \cdot \xi_n^s + h$. $\deg_{\xi_n} h < s$.

Since $\overline{c_{\overline{\nu}}} \neq 0$. $\therefore \sigma(f)$ is ξ_n-dist of order s.

In general, choose a t large enough to work for all f_i.

Thm. 8. (*Weierstrass Division*). Let $g \in T_n$ be a ξ_n-distinguished of some order s. Then for any $f \in T_n$, $\exists!\, q \in T_n$ $r \in T_{n-1}[\xi_n]$ of $\deg_{\xi_n} < s$ s.t. $f = qg + r$. and $|f| = \max(|q|\,|g|, |r|)$.

Pf. Uniqueness. $|f| = \max\{|q|\,|g|, |r|\}$: Assume $f = qg + r$.

WLOG, $|g| = 1$. $\max\{|q|\,|g|, |r|\} = 1$. $\therefore |f| \le \max\{|q|\,|g|, |r|\} = 1$.

If $|f| < 1$, $\Rightarrow \overline{q}\,\overline{g} + \overline{r} = 0$. ($\overline{q} \neq 0$ or $\overline{r} \neq 0$). wich contradicts with Euclid's division in $k[\xi_1, \cdots, \xi_{n-1}][\xi_n]$. $\therefore |f| = 1$. uniqueness is also clear.

Existence: denote $g=\sum_{\nu\ge 0} g_\nu\cdot\xi_n^\nu$, $g_\nu\in T_{n-1}$, $g_s\in T_{n-1}^*$, when $\nu>s$, $|g_\nu|<|g_s|=|g|=1$.

Set $\varepsilon=\max_{\nu>s}|g_\nu|$. $\varepsilon<1$. We show a weaker assertion:

(W). For any $f\in T_n$, $\exists\, q, f_1\in T_n$, $r\in T_{n-1}[\xi_n]$ of $\deg_{\xi_n}<s$ s.t.

$f=qg+r+f_1$ ($|q|,|r|\le|f|$, $|f_1|\le\varepsilon|f|$).

(Let $f_0=f$, $f_i=q_ig+r_i+f_{i+1}$ $\Rightarrow$ $f=(\sum q_i)g+(\sum r_i)$).

Pf of (W). Assume $f=\sum_{m\ge 0} f_m$, $f_m\in T_{n-1}[\xi_n]$, $|f_m|\to 0$. Then only need to prove (W) for f_m. WLOG assume $f\in T_{n-1}[\xi_n]$.

Set $g'=\sum_{i=0}^{s} g_i\xi_n^i$, $f=q\cdot g'+r$ ($\deg_{\xi_n} r<s$). $|f|=\max\{|q|,|r|\}$.

$\therefore f=qg+r+\underbrace{(qg'-qg)}_{f_1}$, $|g'-g|=\varepsilon$, $|q|\le|f|$. $\therefore |f_1|\le\varepsilon\cdot|f|$. □

Cor. 9. Let $g\in T_n$ be ξ_n-dist. of order s. $\exists!$ monic $w\in T_{n-1}[\xi_n]$ dist of order s, $|w|=1$ s.t. $g=ew$ for a $e\in T_n^*$. (w: Weierstrass poly.)

Pf. Existence: $\xi_n^s=qg+r$, $|\xi_n^s|=1=\max\{|q||g|,|r|\}$. $\therefore |r|\le 1$. ($\deg_{\xi_n} r<s$).

Set $w=\xi_n^s-r$. $|w|=1$, dist. of order s.

Suffices to prove $q\in T_n^*$. Assume $|q|=|g|=1$.

$\bar{w}=\bar{q}\bar{g}$ $\Rightarrow$ $\bar{q}\in k^*$ $\therefore q\in T_n^*$.

($\bar{w}$, $\bar{g}$: $\deg_{\xi_n}=s$)

Uniqueness: Assume $g=ew$, let $r=\xi_n^s-w$ $\therefore \xi_n^s=e^{-1}g+r$

$\Rightarrow$ e, r are unique (by Thm 8).

Cor 10. $T_1=k\langle\xi_1\rangle$ is an ED.

pf. Any $g \in T_1$ is $\mathfrak{z}_1$-dist. of some order s. Thm 8 says: $g \to s$ is a well defined Euclidean function.

Cor 11. (Noether Normalization)

For any proper $a \subsetneq T_n$, $\exists$ k-alg. mono. $T_d \hookrightarrow T_n$ s.t. $T_d \to T_n \to T_n/a$ is finite mono. d is uniquely determined, and is called krull dim of T_n/a.

Pf. Choose $g \neq 0 \in a$. Use Lem. 7 to make g $\mathfrak{z}_n$-dist. of some order $s > 0$. By Weierstrass division, $T_n/(g)$ is a free T_{n-1}-module with basis $1, \mathfrak{z}_n, \cdots, \mathfrak{z}_n^{s-1}$. $\therefore$ $T_{n-1} \to T_n \to T_n/(g)$ is finite, mono.

$T_{n-1} \to T_n/g \to T_n/a$, with $\Theta: T_{n-1} \to T_n/a$ the composite; denote $a_1 = \ker \Theta$. If $a_1 = 0$ ✓.

If $a_1 \neq 0$, proceed with a_1 and T_{n-1}. after finite steps, we get finite mono: $T_d \to T_n/a$.

(By prop. 17, $\dim T_n = n$. $\therefore \dim T_n/a = \dim T_d = d$).

Cor 12. For any $m \in \mathrm{Max}\, T_n$, T_n/m is finite over K.

Pf. $\exists d$ s.t. $T_d \hookrightarrow T_n/m$ is finite. $\therefore d = 0$. $\therefore$ T_n/m is finite over $K = T_0$.

Cor 13. $B^n(\bar{K}) \to \mathrm{Max}\, T_n$ is surj

$x \to m_x = \{f \mid f(x) = 0\}$.

Pf. Clearly $m_x \in \mathrm{Max}\, T_n$: $T_n \xrightarrow{\theta_x} K(x_1, \cdots, x_n)$ (where $x = (x_1, \cdots, x_n)$), $\ker \theta_x = m_x$.

$\mathfrak{z}_i \to x_i$

Surj: If $m \in \text{Max}\, T_n$, then T_n/m is finite over K.

$$T_n/m \overset{\varphi}{\hookrightarrow} \bar{K} \quad (T_n \to T_n/m \to \bar{K}). \quad \varphi(\zeta_i) \triangleq x_i \in \bar{K}.$$

$\therefore x = (x_1, \cdots, x_n) \in \bar{K}^n$. Suffices to prove $|x_i| \le 1$ $(\forall i)$.

For any x_i, let $p(\eta) = \eta^r + c_1 \eta^{r-1} + \cdots + c_r$ be its minimal poly over K, let α_j $(j = 1, \cdots, n)$ be its roots. $(\alpha_1 = x_i)$.

$\therefore |\alpha_j| = |\alpha_1|$ $(\forall j)$ If $|x_i| > 1$, then $|c_i| \le |\alpha_1|^i < |\alpha_1|^r = c_r$ $(i < r)$.

$\therefore p(\zeta_i)$ is a unit (Cor 4). Contradicts with $\varphi(p(\zeta_i)) = 0$!

Rmk It's not inj. Example: $O_{\bar{K}} \to \text{Max}\, T_1$ is not inj. Since we can pick any $t \in O_{\bar{K}} \setminus K$, then $m_t = (f_t)$ (f_t: minimal poly of t). and other roots of f_t gives the same maximal ideal.