Proof of the Proper Mapping Theorem

Recall:

X, Y. rigid K-spaces. $\varphi: X \to Y$ is proper if.

(i). φ is separated (i.e. $\Delta: X \to X \times_Y X$ is a closed immersion).

(ii) There exist an admissible affinoid covering $(Y_i)_{i \in I}$, of Y. and for each $i \in I$, two finite admissible affinoid coverings $(X_{ij})_{j=1,\dots,n_i}$ $(X'_{ij})_{j=1,\dots,n_i}$, of $\varphi^{-1}(Y_i)$. s.t. $X_{ij} \Subset_{Y_i} X'_{ij}$ for all i, j.

Thm. (Proper Mapping Theorem).

$\varphi: X \to Y$. proper. $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then $R^q\varphi_*(\mathcal{F})$ $q \geq 0$. are coherent $\mathcal{O}_Y$-modules.

Sketch of proof:

It is local on Y, thus we may assume Y is affinoid, and there exists two finite admissible affinoid coverings $\mathcal{U} = (U_i)_{i=1,\dots,s}$. $\mathcal{V} = (V_i)_{i=1,\dots,s}$, of X, s.t. $V_i \Subset_Y U_i$.

Want to show: $H^q(X, \mathcal{F})$. is a finite module over $B = \mathcal{O}_Y(Y)$

Tate acyclicity $\Rightarrow$ the canonical morphisms

$$H^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{res}} H^q(\mathcal{V}, \mathcal{F}) \to H^q(X, \mathcal{F}). \quad q \geq 0$$

are isoms.

"slogan: isom + compact $\Rightarrow$ finite dim".

Let $Z^q(\mathcal{V}, \mathcal{F}) = \ker(d: C^q(\mathcal{V}, \mathcal{F}) \to C^{q+1}(\mathcal{V}, \mathcal{F}))$.

It is enough to show:

$$f^q: C^{q-1}(\mathcal{V}, \mathcal{F}) \to Z^q(\mathcal{V}, \mathcal{F}) \quad q \geq 0$$

have finite B-modules as cokernels.

Let $$r^q : Z^q(\mathcal{U},\mathcal{F}) \longrightarrow Z^q(\mathcal{V},\mathcal{F})$$
be the restristion map. Since $H^q(\mathcal{U},\mathcal{F}) \cong H^q(\mathcal{V},\mathcal{F})$,
$$f^q + r^q : C^{q-1}(\mathcal{V},\mathcal{F}) \oplus Z^q(\mathcal{U},\mathcal{F}) \longrightarrow Z^q(\mathcal{V},\mathcal{F})$$
is surjective.

" $f^q + r^q$ surj. + r^q : "compact" $\Rightarrow$ f^q has finite cokernel."

B = affinoid K-alg. equipped with a fixed residue norm $|\cdot|$.

For complet normed B-module M. set
$$M^\circ = \{ x \in M,\ |x|_M \leq 1 \}.$$
For $f : M \longrightarrow N$. B-linear, conts, set
$$|f| = \sup\left\{ \frac{|f(x)|_N}{|x|_M} ;\ x \in M - \{0\} \right\}.$$
It is a cplt B-module norm on $\mathrm{Hom}^{\text{conts}}_{B\text{-linear}}(M,N)$.

Def. A conts B-linear homomorphism $f : M \to N$ is called <u>completly continous</u> if it is the limit of a sequence $(f_i)_{i \in \mathbb{N}}$ of conts B-linear homomorphisms s.t. $\mathrm{im}(f_i)$ is a finite B-module. for all $i \in \mathbb{N}$.

Furthermore if there is an element $c \in R - \{0\}$. (R = valuation ring of K) s.t. $\forall i \in \mathbb{N}$, $c f_i(M^\circ)$ is contained in a finite B°-submodule of N°, which may depend on i, then f is called strictly completly continous.

Prop./E.g. Let $f : B\langle\zeta\rangle \to A$ be a K-homomorphism where A, B are affinoid K-alg, $\zeta = (\zeta_1, \cdots \zeta_n)$. a system of variables. Consider $B\langle\zeta\rangle$. the Gauß norm derived from a given residue norm on B. and on A any residue norm s.t. $f|_B : B \to A$ is contractive.

(i.e. $|f(b)| \le |b|$). Then if $|f(\xi_i)|_{\sup} < 1$, $\forall i$, f is strictly completely continuous.

Pf:

$$|f(\xi_i)|_{\sup} < 1 \overset{3.1.18}{\Longrightarrow} f(\xi_i) \text{ is topological nilpotent.}$$

Let $M_i = \bigoplus_{|\nu| = i} B\xi^\nu$ then $M = \hat{\bigoplus}_{i \in \mathbb{N}} M_i$. let $f_i : M \to A$ s.t. $f_i|_{M_i} = f|_{M_i}$, $f_i|_{\hat{\bigoplus}_{j \ne i} M_j} = 0$. Then $f_i(M)$ is a finite rank B-module and $f = \sum_{i \in \mathbb{N}} f_i \Rightarrow f$ is completly conts.

Choosing $c \in R - \{0\}$ s.t. $|f(\xi^\nu)| \le |c|^{-1}$, $\forall \nu$, we get $cf_i(M^\circ) \subset A^\circ$. Since $f_i(M^\circ) = f_i(M_i^\circ)$ is a finite B°-module, f is strictly completely conts. $\square$

Thm. Let $f, g : M \to N$ be conts homomorphisms of cplt normed B-modules. Assume

(i) f is surjective.

(ii) g is completly conts

Then $\operatorname{im}(f+g)$ is closed in N and $N/\operatorname{im}(f+g)$ is a finite B-module.

Pf:

1°. $|g|$ is "small":

By Banach's Theorem, f is open $\Rightarrow \exists t \in K^*$, s.t. $tN^\circ \subset f(M^\circ)$

$\Rightarrow$ Replacing t by ct for $c \in K^*$, $|c| < 1$, we see $\forall y \in N$, $\exists x \in M$ s.t. $f(x) = y$, $|x| \le |t|^{-1}|y|$.

Now consider the case $|g| = \alpha |t|$, $\alpha < 1$. Claim: $f+g$ is still surj

Given $y \in N - \{0\}$, we can pick $x \in M$, s.t. $f(x) = y$, $|x| \le |t|^{-1}|y|$

Let $y' = (f+g)(x) - y = g(x)$ Then $|y'| \le \alpha \cdot |y|$

Proceeding with y' in the same way + limit process $\Rightarrow$ $f+g$ is surj.

$\forall y \in N$, chose $c' \in K^*$ s.t. $|c'y| \leq |t|$ $\Rightarrow \exists x \in M^\circ$, $f(x) = c'y$

$\underset{c^n,\, n \in \mathbb{Z}}{\overset{\shortparallel}{c'}}$ $\quad |c||t| <$

2°. g has finite image. Then $M/\ker g$ is a finite B-mod.

Consider

$$\begin{tikzcd} M \arrow[r, two heads, "f"] \arrow[d] & N \arrow[d, "p"] \\ M/\ker g \arrow[r, "\bar{f}"] & N/f(\ker g) \end{tikzcd}$$

Then $N/f(\ker g)$ is a finite B-module

$$f(\ker g) = (f+g)(\ker g) \subset (f+g)(M).$$

$\Rightarrow N/(f+g)(M)$ is a finite B module

To show $\operatorname{im}(f+g)$ is closed in N. Since $\ker g$ is closed in M, we can provide $M/\ker g$ with residue norm from M.

Using 2.3.10. any submodule of such a finite B-module is closed

Thus. $\ker \bar{f}$ is closed. and we can consider the residue norm via $\bar{f}$ on $N/f(\ker g)$.

By Banach's thm, the norm of N coincide with the residue norm via f. Thus $N \to N/f(\ker g)$ is conts.

Since $(f+g)(M)$ is the inverse of a submodule of $N/f(\ker g)$.

any any such submodule is closed. $\Rightarrow (f+g)(M)$ is closed in N $\square$

$\exists\, g'$. s.t $\operatorname{im} g'$ is finite and $|g-g'| < |t|^{-1}$

f is surj $\overset{①}{\Longrightarrow}$ $f + (g-g')$ is surj. $\overset{②}{\Longrightarrow}$ $\operatorname{Im} f+g$ ...

$\square$

In our situation, we have maps.

$$f^q + r^q : C^{q-1}(\mathcal{V}, \mathcal{F}) \oplus Z^q(\mathcal{U}, \mathcal{F}) \twoheadrightarrow Z^q(\mathcal{V}, \mathcal{F}) \quad q \geq 0.$$

We want to apply the thm to $f = f^q + r^q$, $g = -r^q$.

But we don't know if g is completly conts:

$V_i \Subset_Y U_i$

$\Rightarrow \exists\ B\langle\xi\rangle \twoheadrightarrow \mathcal{O}_X(U_i)$ s.t. $B\langle\xi\rangle \twoheadrightarrow \mathcal{O}_X(U_i) \to \mathcal{O}_X(V_i)$, $\quad \underbrace{B\langle\xi\rangle \to \mathcal{O}_X(V_i)}_{p}$

$|\xi_i|_{sup} < 1$ in $\mathcal{O}_X(V_i)$. p is completely conts

$f: M \to N$. completely conts.

Q: $f|_{f^{-1}(N')}: f^{-1}(N') \to N'$. $\quad N' \subset N$ closed

completly conts

$P \overset{p}{\twoheadrightarrow} M \xrightarrow{f} N$ completely conts.

Q: f is ... ?

We need a generalization of the Thm:

Thm. Let $f, g: M \to N$, as above. Assume

(i) f is surj.

(ii) g is part of a sequence $M^b \overset{p}{\twoheadrightarrow} M \xrightarrow{g} N \overset{j}{\hookrightarrow} N^\#$ of conts morphisms of cplt normed B-modules where p is an epimorphism. j. identifies N with a closed submodule of $N^\#$. and $j \circ g \circ p$ is strictly completly conts.

Then $\mathrm{im}(f+g)$ is closed in N, and coker $N/\mathrm{im}(f+g)$ is a finite B-module.

Lem 1. E be a finite B°-module, $E' \subset E$ a B°-submodule. Then for any $0 < \alpha < 1$. $\exists$ a finite B°-submodule $E'' \subset E'$ s.t. $aE' \subset E''$ $\forall a \in R$, $|a| \le \alpha$.

Pf: Let $\pi_n : T_n = K\langle\xi\rangle \longrightarrow B$. be an epimorphism defining the residue norm on B. Then the induced morphism $\pi^\circ : T_n^\circ = R\langle\xi\rangle \to B^\circ$ is surj. If R is a DVR. $R\langle\xi\rangle$. is Noetherian, then we can take $E'' = E'$.

If the valuation on K is not discrete. Note that

· Let $0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$ be an exact seq, of finite B°-modules. then the lemma holds for E if and only if it holds for E_1, E_2. (ex) $B = K\langle\xi\rangle$ and

· Then we reduce to E is a finite free $R\langle\xi\rangle$-mod.

$\rightsquigarrow$ reduce to $E = R\langle\xi\rangle$. Then E' is an ideal in $R\langle\xi\rangle$.

Then use induction on n, the number of variables. Assume $n > 0$, $E' \neq 0$. Let $\beta = \sup\{|h| ; h \in E'\}$, take $g \in E'$. s.t. $|g| > \alpha\beta$.

$\exists\, c \in R$ s.t. $|c| = |g|$. Let $f = c^{-1}g$ then $|f| = 1$. After change of variables, we may assume f is ξ_n-distinguished of order $s \geq 0$. Then by Weierstrass division, $R\langle\xi\rangle/(f)$ is a finite $R\langle\xi'\rangle$-mod where $\xi' = (\xi_1, \cdots \xi_{n-1})$ and

$$0 \longrightarrow (f) \longrightarrow R\langle\xi\rangle \longrightarrow R\langle\xi\rangle/f \longrightarrow 0$$

By induction, it suffices to show the lemma holds for $E_1' = E' \cap (f) \subset (f)$ but now $a E_1' \subset (g)$. , so we can take $E_1'' = (g)$. □

Lem 2.

Let $M \xrightarrow{g} N \xhookrightarrow{j} N^\#$ be a morphism of cplt normed B-modules. where j identifies N with a closed submodule of $N^\#$. Assume M is topologically free. in the sense that $\exists (e_\lambda)_{\lambda\in\Lambda}$ of elements in M st. $\forall x \in M$ can be uniquely written as a converging

Series $x = \sum_{\lambda \in \Lambda} b_\lambda e_\lambda$ with $b_\lambda \in B$, and $|x| = \max_{\lambda \in \Lambda} |b_\lambda|$ $(|b_\lambda| \to 0)$

Then, if $j \circ g$ is strictly completly conts, the same is true for g

pf:

We may assume $j \circ g$, g are contractive. Then g, j restrict to $M^\circ \longrightarrow N^\circ \hookrightarrow N^{\#\circ}$.

Since $j \circ g$ is strictly completly conts, $\exists\ h_i : M \longrightarrow N^\#$ $i \in \mathbb{N}$, s.t. $j \circ g = \lim_{i \in \mathbb{N}} h_i$, $\exists\ c \in R \setminus \{0\}$ such that $c h_i(M^\circ)$ is contained in a finite B°-submodule of $N^{\#\circ}$. Adjusting norms on N, $N^\#$ by $|c|^{-1}$.. we may assume $c = 1$

Let $h = h_i$ for $i \gg 0$, by lemma 1, $\exists$ finite B°-submodule $E'' \subset h(M^\circ)$ s.t. $a h(M^\circ) \subset E''$, $a \in R$, $0 < |a| < 1$

Fix generators $y_1, \cdots y_r$ of E'', let $x_1, \cdots x_r \in M^\circ$ s.t. $h(x_j) = y_j$

Let $z_j = g(x_j)$ Then we have approximated $y_j \in N^{\#\circ}$ by $z_j \in N^\circ$.

Since $a h(M^\circ) \subset E''$, $\exists\ b_{j\lambda} \in B$. $j = 1, \cdots r$, $\lambda \in \Lambda$ s.t.

$$h(e_\lambda) = \sum_{j=1}^{r} b_{j\lambda} y_j \qquad |b_{j\lambda}| \le |a|^{-1}$$

Then we define $g' : M \longrightarrow N$ by

$$g'(e_\lambda) = \sum_{j=1}^{r} b_{j\lambda} z_j$$

$$\begin{tikzcd} M \arrow[r, dashed, "x_i \mapsto z_i"] \arrow[rr, bend right, "x_i \mapsto y_i"'] & N \arrow[r, hook] & N^\# \end{tikzcd}$$

when $h \to j \circ g$, $g' \to g$.

$|e_\lambda| = 1$

$a h(e_\lambda) \in E''$

$\Rightarrow a \cdot b_{j\lambda} \in B^\circ$

$\Rightarrow |a| \cdot |b_{j\lambda}| \le 1$

Now we can prove the thm:

(i) : $f : M \longrightarrow N$. surj.

(ii). $M^b \overset{P}{\twoheadrightarrow} M \xrightarrow{g} N \overset{\hat{i}}{\hookrightarrow} N^\#$ $\quad j \circ g \circ p$ strictly completly cnts.

Then $\operatorname{im}(f+g)$ is closed, $\operatorname{coker}(f+g)$ is a finite B-mod.

First, note that $p: M^b \to M$ is not really relevant.

$f' = f\circ p : M^b \to N$. $g' = g\circ p$.

$\operatorname{im}(f'+g') = \operatorname{im}(f+g)$

1°. If M^b is topologically free. Lem 2 $\Rightarrow$ $g\circ p$ is strictly cplt cnts.

2°. In general, consider a bounded generating system $(x_\lambda)_{\lambda\in\Lambda}$ for M^b, let M^{bb} be the completion of $B^{(\Lambda)}$, the free B-mod gen. by Λ. Then $M^{bb} \twoheadrightarrow M^b$ and M^{bb} is top. free. $\square$

Prop. Let $\varphi: X \to Y$ be a proper morphism of rigid K-spaces, where Y is affinoid, $\exists$ two admissible affinoid coverings $\mathcal{U} = (U_i)_{i=1,\dots,s}$, $\mathcal{V} = (V_i)_{i=1,\dots,s}$ of X s.t. $V_i \Subset_Y U_i$. $\forall i$.

Then $H^q(X,\mathcal{F})$ is a finite module over $B = \mathcal{O}_Y(Y)$. $q \geq 0$.

pf:

Since $V_i \Subset_Y U_i$, there is an epimorphism

$$E_i = B\langle \xi_1, \dots, \xi_n\rangle \twoheadrightarrow \mathcal{O}_X(U_i)$$

s.t. $|\xi_j|_{\sup} < 1$ in $\mathcal{O}_X(V_i)$. Then $E_i \to \mathcal{O}_X(V_i)$ is strictly completely cnts. Thus:

$\forall q \in \mathbb{N}$, $\exists$ topologically free cplt normed B-mod E^q with cnts epi. $p: E^q \to C^q(\mathcal{U},\mathcal{F})$. s.t.

$$E^q \xrightarrow{p} C^q(\mathcal{U},\mathcal{F}) \xrightarrow{\text{res}} C^q(\mathcal{V},\mathcal{F})$$

is strictly cpltly cnts.

Recall. we have

$$f^q: C^{q-1}(\mathcal{V},\mathcal{F}) \longrightarrow Z^q(\mathcal{V},\mathcal{F}). \qquad q\geq 0$$

and

$$f^q+r^q: C^{q-1}(\mathcal{V},\mathcal{F}) \oplus Z^q(\mathcal{U},\mathcal{F}) \twoheadrightarrow Z^q(\mathcal{V},\mathcal{F}) \qquad q\geq 0$$

and the composition

$$C^{q-1}(\mathcal{V},\mathcal{F}) \oplus p^{-1}(Z^q(\mathcal{U},\mathcal{F})) \xrightarrow{id\times p} C^{q-1}(\mathcal{V},\mathcal{F}) \oplus Z^q(\mathcal{U},\mathcal{F}) \xrightarrow{r^q=(0,res)} Z^q(\mathcal{V},\mathcal{F}) \xhookrightarrow{j} C^q(\mathcal{V},\mathcal{F})$$

is strictly completely conts. Thus by the above thm. coker of $f^q = (f^q+r^q) - r^q$ is a finite B-mod $\square$

Now let's prove $R^q\varphi_*\mathcal{F}$ is the sheaf associated to $H^q(X,\mathcal{F})$:

We want to show for $Y' = \operatorname{Sp} B'$ an affinoid subdomain in Y. Let $X' = X\times_Y Y'$. then

$$H^q(X,\mathcal{F}) \otimes_B B' \cong H^q(X',\mathcal{F}).$$

step 1.

Thm (Theorem on formal functions).

$\varphi: X \to Y$, $\mathcal{F}$ as above. $b\in B = \mathcal{O}_Y(Y)$. Then the canonical morphism

$$\varprojlim_i H^q(X,\mathcal{F})/b^i H^q(X,\mathcal{F}) \longrightarrow \varprojlim_i H^q(X,\mathcal{F}/b^i\mathcal{F}).$$

is an isom. $\forall q\geq 0$ finite B-module.

pf:

The same as the scheme version $\square$

step 2,

Use induction on the Krull dim d of B.

$d>0$. It's enough to show that all localizations

$$H^q(X,\mathcal{F})\otimes_B B'_{\mathfrak{m}'} \longrightarrow H^q(X',\mathcal{F})\otimes_{B'} B'_{\mathfrak{m}'}$$

at maximal ideals $\mathfrak{m}'\subset B'$ are isom.

Since the $\mathfrak{m}'$-adic completion $\hat{B}'_{\mathfrak{m}'}$ of $B'_{\mathfrak{m}'}$ is faithfully flat over $B'_{\mathfrak{m}'}$, suffice to show

$$H^q(X,\mathcal{F})\otimes_B \hat{B}'_{\mathfrak{m}'} \longrightarrow H^q(X',\mathcal{F})\otimes_{B'} \hat{B}'_{\mathfrak{m}'}$$

is isom.

By 3.3.10, $\exists$ maximal ideal $\mathfrak{m}\subset B$ s.t. $\mathfrak{m}'=\mathfrak{m}B'$.
Take $b\in\mathfrak{m}$ s.t. B/b^i has Krull dim $<d$ $\forall i\in\mathbb{N}$.

Let $Y_i=\mathrm{Sp}\,B/b^i$. Then by induction ($H^q(X\times_Y Y_i, \mathcal{F}/b^i\mathcal{F})\simeq H^q(X, \mathcal{F}/b^i\mathcal{F})$)

$$H^q(X,\mathcal{F}/b^i\mathcal{F})\otimes_B B' \simeq H^q(X',\mathcal{F}/b^i\mathcal{F}).$$

Taking $\varprojlim$.

$$\begin{array}{ccc}
\varprojlim_i \left(H^q(X,\mathcal{F}/b^i\mathcal{F})\otimes_B B'\right) & \simeq & \varprojlim_i H^q(X',\mathcal{F}/b^i\mathcal{F}) \\
\wr\wr & & \wr\wr \\
\underline{H^q(X,\mathcal{F})\otimes_B \hat{B}'} & & \underline{H^q(X',\mathcal{F})\otimes_{B'} \hat{B}'}
\end{array}$$

where $\hat{B}'$ is the b-adic completion of B'

Fact: the b-adic completion of a <u>finite</u> B'-mod. M' is $M'\otimes_B \hat{B}'$.

Since $b\in\mathfrak{m}'$, $B'\longrightarrow \hat{B}'_{\mathfrak{m}'}$ factors through $\hat{B}'$

Thus. $$H^q(X,\mathcal{F})\otimes_B \overset{\mathfrak{m}'\text{-adic}}{\hat{B}'_{\mathfrak{m}'}} \xrightarrow{\sim} H^q(X',\mathcal{F})\otimes_{B'} \overset{b\text{-adic}}{\hat{B}'_{\mathfrak{m}'}}. \qquad \square$$