2023 Winter – Seminar on *Formal and Rigid Geometry*

Talk 3-4

AFFINOID ALGEBRAS AND THEIR ASSOCIATED SPACES

This is the live-TeXed notes by Wenhan Dai for this seminar. The note-taker claims no originality and takes full responsibility for all errors made therein.

- **Talk 3 (Affinoid algebras and affinoid spaces):** Cover [Bos14, §3.1-3.2]. State some immediate consequences from last chapter for affinoid algebras (Propositions 3-5). Then discuss the residue norm and the supremum norm and their relations, in particular, prove Theorem 17 and show that all residue norms are equivalent (Proposition 20). Finally, introduce the affinoid spaces (§3.2).
- **Talk 4 (Affinoid subdomains):** Cover [Bos14, §3.3]. Understand the canonical topology of affinoid spaces and their affinoid subdomains, in particular, prove Proposition 11. Moreover, discuss some further properties about affinoid subdomains: Proposition 12-Theorem 20.

1. Affinoid algebra

Fix a non-archimedean field K with a valuation $|\cdot|$.

Definition 1.1. A K-algebra A is called an *affinoid K-algebra* if there is a K-algebra epimorphism

$$\alpha : T_n \longrightarrow A$$

for some integer $n \in \mathbb{N}$. Namely, A can be realized as a quotient algebra of some Tate algebra.

Notation 1.2. Denote $\mathcal{A}$ the category of affinoid K-algebras. Its objects are affinoid K-algebras and the morphisms are given by K-algebra homomorphisms.

Proposition 1.3. *Let $R, A_1, A_2 \in \mathcal{A}$ with $\varphi : R \to A_1$ and $\psi : R \to A_2$ two K-algebra homomorphisms. Then there exists a unique $T \in \mathcal{A}$ such that the following diagram commutes:*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & A_1 \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ A_2 & \longrightarrow & T \end{array}$$

and satisfies the following universal property. For another base algebra $D \in \mathcal{A}$ with $D \to A_1$ and $D \to A_2$ homomorphisms of K-algebras, there is another K-algebra homomorphism $T \to D$ such that the diagram commutes:

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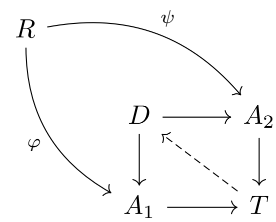


In fact, T can be constructed as the complete tensor product[1]

$$T = A_1 \widehat{\otimes}_R A_2.$$

We list out some basic properties of affinoid algebras.

Proposition 1.4. *Let $A \in \mathcal{A}$. Then A is noetherian, Jacobson, and there is a finite monomorphism $T_d \hookrightarrow A$ for some integer $d \in \mathbb{N}$ (namely, it admits Noether normalization).*

Proposition 1.5. *Let $A \in \mathcal{A}$ and $\mathfrak{q} \lhd A$ an ideal such that $\sqrt{\mathfrak{q}} \in \operatorname{Max} A$ is a maximal ideal. Then $A/\mathfrak{q}$ is a finite-dimensional K-vector space.*

Proof. Let $\mathfrak{m} = \sqrt{\mathfrak{q}} \in \operatorname{Max} A$. Then there is an integral homomorphism

$$T_d \hookrightarrow A/\mathfrak{q} \twoheadrightarrow A/\mathfrak{m}.$$

Since $A/\mathfrak{m}$ is a field, it forces T_d to be a field with $d = 0$, or equivalently $T_d = K$. □

Definition 1.6. Given any affinoid algebra A with an epimorphism $\alpha : T_n \twoheadrightarrow A$, denote $\overline{f} = \alpha(f)$ the residue of f. Also define the map $|\cdot|_\alpha : A \to \mathbb{R}_{\geqslant 0}$ via

$$|\overline{f}|_\alpha := \inf_{a \in \ker \alpha} |f - a| = \inf_{g \in T_n,\ \alpha(g) = \overline{f}} |g|.$$

Proposition 1.7. *Let $\mathfrak{a} \lhd T_n$ be an ideal and $A = T_n/\mathfrak{a}$. Fix $\alpha : T_n \to A$ as above. Then for $(A, |\cdot|_\alpha)$:*

(1) *$(A, |\cdot|_\alpha)$ is a K-algebra norm, i.e. it is a ring norm and a K-vector space norm, such that $|x| = 0$ if and only if $x = 0$, $|f + g| \leqslant \max\{|f|, |g|\}$, $|fg| \leqslant |f| \cdot |g|$ for all $f, g \in T_n$, $|1| \leqslant 1$, and $|cf| = |c| \cdot |f|$ for each $c \in K$. Moreover, $\alpha : T_n \to T_n/\mathfrak{a}$ is continuous and open, and $|\cdot|_\alpha$ induces the quotient topology of T_n on $T_n/\mathfrak{a}$.*
(2) *$T_n/\mathfrak{a}$ is complete under $|\cdot|_\alpha$.*
(3) *For any $\overline{f} \in T_n/\mathfrak{a}$, there is $f \in T_n$ such that $\alpha(f) = \overline{f}$, and $|\overline{f}|_\alpha = |f|$. In particular, $|(T_n/\mathfrak{a})^*|_\alpha \subseteq |T_n^*|$.*

We have seen that the affinoid algebra A is defined by an ideal $\mathfrak{a}$ in T_n, together with a choice of $\alpha : T_n \to A$, which is not necessarily the canonical projection of K-algebras. On the other hand, the property of topology on A seems to depend strongly on the choice of α. We will see later that the affinoid topology is well-defined, i.e. insensitive with respect to the choice of α. However, there is still a long way from this conclusion.

Definition 1.8. Define the supreme norm

$$|f|_{\sup} := \sup_{x \in \operatorname{Max} A} |f(x)|,$$

[1]For more details, see [Bos14, Appendix B].

where $f(x)$ is given as follows. For each $x \in \operatorname{Max} A$, A/x is a finite extension of K. Denote $f(x)$ the residue class of f in A/x.

Actually, $|\cdot|_{\sup}$ is not a norm (even if it is named with "norm"), and it is only a semi-norm, for which $|f|_{\sup} = 0$ does not imply that $f = 0$.

Proposition 1.9. *Let A be an affinoid algebra.*

1. *$|\cdot|_{\sup}$ is power multiplicative, i.e. for all $f \in A$ we have $|f^n|_{\sup} = |f|_{\sup}^n$.*
2. *Let $\varphi : B \to A$ be a morphism between affinoid K-algebras. Then $|\varphi(b)|_{\sup} \leqslant |b|_{\sup}$ for all $b \in B$.*

Proof. We only prove (2). For all $\mathfrak{m} \in \operatorname{Max} A$, we have

$$K \hookrightarrow B/\varphi^{-1}(\mathfrak{m}) \hookrightarrow A/\mathfrak{m}.$$

Hence $\varphi^{-1}(\mathfrak{m}) \in \operatorname{Max} B$. And

$$|b(\varphi^{-1}(\mathfrak{m}))| = |\varphi(b)(\mathfrak{m})| \leqslant |b|_{\sup}.$$

This is the desired inequality. □

Proposition 1.10. *On T_n, $|\cdot|_{\sup} = |\cdot|$, the usual Gauss norm on T_n.*

Proof. By the maximum principle,

$$|f| = \sup\{|f(x)| : x \in \mathbb{B}^n(\overline{K})\}$$

with a surjective map $\mathbb{B}^n(\overline{K}) \twoheadrightarrow \operatorname{Max} T_n$. Hence

$$\sup\{|f(x)| : x \in \mathbb{B}^n(\overline{K})\} = \sup\{|f(\mathfrak{m})| : \mathfrak{m} \in \operatorname{Max} T_n\}.$$

□

Proposition 1.11. *Let $A \in \mathcal{A}$ with $\alpha : T_n \twoheadrightarrow A$. Then for each $f \in A$,*

$$|f|_{\sup} \leqslant |f|_\alpha.$$

In particular, since $|\cdot|_\alpha$ is defined by the quotient and is finite, $|\cdot|_{\sup}$ is finite.

Proof. For each $\mathfrak{m} \in \operatorname{Max} A$, we have

$$K \hookrightarrow T_n/\alpha^{-1}(\mathfrak{m}) \to A/\mathfrak{m}.$$

For any $f \in A$ there is $g \in T_n$ such that $|f|_\alpha = |g|$. And hence

$$|f(\mathfrak{m})| = g(\alpha^{-1}(\mathfrak{m}))| \leqslant |g| = |f|_\alpha.$$

This leads to $|f|_{\sup} \leqslant |f|_\alpha$. □

Proposition 1.12. *Let $A \in \mathcal{A}$ with $f \in A$. Then*

$$|f|_{\sup} = 0 \iff f \text{ is nilpotent.}$$

Proof. Assume $|f|_{\sup} = 0$. Then equivalently,

$$|f(\mathfrak{m})| = 0 \text{ for all } \mathfrak{m} \in \operatorname{Max} A \iff f \in \bigcap_{\mathfrak{m} \in \operatorname{Max} A} \mathfrak{m} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{0}$$
$$\iff f \text{ is nilpotent.}$$

The intersection of $\mathfrak{p} \in \operatorname{Spec} A$ equals to zero as A is Jacobson. □

Here comes the most crucial result that not only Tate algebras satisfy the maximum principle but also affinoid algebras do.

Theorem 1.13 (Maximum principle)**.** *Let $A \in \mathcal{A}$ with $f \in A$. Then there is $x \in \operatorname{Max} A$ such that*

$$|f|_{\sup} = |f(x)|.$$

The proof of Theorem 1.13 has become more complicated and subtle than the previous version, so that some lemmas are in need.

Lemma 1.14. *For all polynomial $p(\zeta) \in K[\zeta]$, write*

$$p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r = \prod_{i=1}^{r}(\zeta - \alpha_j), \quad c_i \in K,\ \alpha_i \in \overline{K},$$

we have

$$\max_{j=1,\dots,r} |\alpha_j| = \max_{i=1,\dots,r} |c_i|^{1/i}.$$

Proof. It is straightforward to apply Vieta's theorem. We have

$$|c_i|^{1/i} \leqslant \max_{j=1,\dots,r} |\alpha_j|.$$

On the other hand, assume that $|\alpha_1| = \cdots = |\alpha_s| > |\alpha_{s+1}|, \dots, |\alpha_r|$ for some $1 \leqslant s \leqslant r$. Then

$$|c_s| = \sum_{1\leqslant i_1<\cdots<i_s\leqslant r} \alpha_{i_1}\cdots\alpha_{i_s} = |\alpha_1\cdots\alpha_s| = \left(\max_{j=1,\dots,r} |\alpha_j|\right)^s.$$

Therefore, $|c_s|^{1/s} = \max_{j=1,\dots,r} |\alpha_j|$. □

Now take $p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r$ with $c_i \in A$. Recall that A is a semi-normed ring. Define

$$\sigma(p) = \max_{i=1,\dots,r} |c_i|^{1/i},$$

which is called the *spectral value* of p.

Lemma 1.15. *For $p, q \in A[\zeta]$ with p, q monic, we have*

$$\sigma(pq) \leqslant \max\{\sigma(p), \sigma(q)\}.$$

The following result is more important.

Lemma 1.16. *Let $T_d \hookrightarrow A$ be a finite homomorphism of K-algebras, where A is torsion-free with $f \in A$. Then:*

(1) *There is a unique monic polynomial*

$$p_f = \zeta^r + a_1\zeta^{r-1} + \cdots + a_r \in T_d[\zeta], \quad a_i \in T_d$$

of minimal degree such that $p_f(f) = 0$. Moreover, set the homomorphism

$$\varphi : T_d[\zeta] \longrightarrow A, \quad \zeta \longmapsto f.$$

Then $(p_f) = \ker\varphi$ as a principal ideal of $T_d[\zeta]$.

(2) *Fix* $x \in \operatorname{Max} T_d$. *Let* $y_1, \dots, y_s \in \operatorname{Max} A$ *be all of its inverse images along* $T_d \hookrightarrow A$, *i.e. those maximal ideals that restrict to* x *on* T_d. *Then*

$$\max_{i=1,\dots,r} |f(y_j)| = \max_{i=1,\dots,r} |a_i(x)|^{1/i}.$$

(3) *As a consequence of* (2),

$$|f|_{\sup} = \max_{i=1,\dots,r} |a_i|_{\sup}^{1/i}.$$

Proof. (1) Denote $F = \operatorname{Frac} T_d$. Then the diagram

$$\begin{array}{ccc} T_d & \hookrightarrow & A \\ \downarrow & & \downarrow \\ F & \hookrightarrow & F \otimes_{T_d} A \end{array}$$

commutes. And the bottom horizontal map is induced from $T_d \hookrightarrow A$, which further leads to $T_d \cong F \otimes_{T_d} T_d \hookrightarrow F \otimes_{T_d} A$ as F is flat. The right vertical map induces

$$A \otimes_{T_d} T_d \hookrightarrow F \otimes_{T_d} A = T_{d,0} \otimes_{T_d} A = A_0$$

by noting that $F = T_{d,0}$ as a localization. Consider

$$F[\zeta] \longrightarrow F \otimes_{T_d} A, \quad \zeta \longmapsto f.$$

Since F is a PID, the kernel of this map is (p_f) for some $p_f \in F[\zeta]$. We show that if p_f is monic then it is unique. Since $T_d \hookrightarrow A$ is an integral homomorphism, there is a monic $h \in T_d[\zeta]$ such that $h(f) = 0$. It follows that $p_f \mid h$ in $F[\zeta]$. Write $h = p_f \cdot s$. If p_f is monic then $p_f \in T_d[\zeta]$.

(2) For $f \in A$ we have integral extensions

$$T_d \hookrightarrow T_d[f] \hookrightarrow A$$

which induces a surjective composite

$$\operatorname{Max} A \twoheadrightarrow \operatorname{Max} T_d[f] \twoheadrightarrow \operatorname{Max} T_d.$$

If $\mathfrak{n}_1, \mathfrak{n}_2 \in \operatorname{Max} A$ such that $\mathfrak{n}_1 \cap T_d[f] = \mathfrak{m} = \mathfrak{n}_2 \cap T_d[f]$, then we have

$$T_d[f]/\mathfrak{m} \hookrightarrow A/\mathfrak{n}_1, \quad T_d[f]/\mathfrak{m} \hookrightarrow A/\mathfrak{n}_2,$$

satisfying $|f(\mathfrak{n}_1)| = |f(\mathfrak{m})| = |f(\mathfrak{n}_2)|$. So it suffices to consider the preimage, say $\mathfrak{m}$, of $x \in \operatorname{Max} T_d$ in $\operatorname{Max} T_d[f]$. Without loss of generality, assume $A = T_d[f]$. Let $L = T_d/x$ with $A \to A/(x)$, which sends f to the residue class $\overline{f}$. By (1) we have $\overline{p}_f \in L[\zeta]$. Then

$$L \longrightarrow A/(x) = L[\zeta]/(\overline{p}_f) = L[\overline{f}]$$

since $\overline{p}_f(\overline{f}) = 0$. Denote $\alpha_1, \dots, \alpha_r$ all the roots of $\overline{p}_f$ in L. According to Lemma 1.14,

$$\max_{i=1,\dots,r} |\alpha_i| = \max_{i=1,\dots,r} |a_i(x)|^{1/i}.$$

And on the other hand, for

$$\varphi_i \in L[\overline{f}] \longrightarrow L[\alpha_i], \quad \overline{f} \longmapsto \alpha_i$$

we have $\{\ker \varphi_i\}_{i=1}^r = \{y_1, \dots, y_s\}$. This renders

$$\max_{i=1,\dots,s} |f(y_i)| = \max_{i=1,\dots,r} |\alpha_i|.$$

(3) It is a consequence of (2).

So we have completed the proof. □

We are to drop the assumption that A is torsion-free on T_n.

Lemma 1.17. *Let $\varphi : B \to A$ be a finite homomorphism of K-algebras. Then for any $f \in A$ there is*

$$f^r + b_1 f^{r-1} + \cdots b_r = 0, \quad b_i \in B$$

such that

$$|f|_{\sup} = \max_{i=1,\dots,r} |b_i|_{\sup}^{1/i}.$$

Proof. The main idea is to reduce the case to that of Lemma 1.16. If A is an integral domain, we will find $T_d \to B$ such that the normalization $T_d \hookrightarrow B/\ker\varphi$, which is a finite monomorphism. Hence there is a composite

$$T_d \hookrightarrow B/\ker\varphi \hookrightarrow A.$$

Hence $T_d \hookrightarrow A$ is a finite monomorphism. As a T_d-module, A is torsion-free. So there is $p_f = \zeta^r + a_1\zeta^{r-1} + \cdots + a_r$ with $p_f(f) = 0$ and $a_i \in T_d$ such that $|f|_{\sup} = \max_{i=1,\dots,r} |a_i|_{\sup}^{1/i}$ by Lemma 1.16.

Consider the homomorphism

$$T_d \longrightarrow B, \quad a_i \longmapsto b_i$$

with $|b_i|_{\sup} \leqslant |a_i|_{\sup}$. It forces

$$|f|_{\sup} \geqslant \max_i |b_i|_{\sup}^{1/i}.$$

And we have $f^r + b_1 f^{r-1} + \cdots + b_r = 0$. On the other hand, by the non-archimedean inequality,

$$|f^r|_{\sup} \leqslant |b_i f^{r-i}|_{\sup}$$

for some i. This shows $|f|_{\sup} \leqslant |b_i|_{\sup}^{1/i}$. Combining these properties, we get

$$|f|_{\sup} = \max_i |b_i|_{\sup}^{1/i}.$$

More generally, if A is not an integral domain, choose minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ such that

$$\operatorname{Max} A = \bigcup_{i=1}^{s} \operatorname{Max} A/\mathfrak{p}_i, \quad |f|_{\sup} = \max_{i=1,\dots,s} |f_i|, \ f_i \in A/\mathfrak{p}_i.$$

Consider the homomorphism $B \to A \to A/\mathfrak{p}_i$, where the second is a finite homomorphism. We see for $i = 1, \dots, s$ there is $q_i \in B[\zeta]$ such that $q_i(f_i) = 0$ and

$$|f_i|_{\sup} = \sigma(q_i) = \max_j |c_{ij}|^{1/j}$$

where c_{ij}'s are coefficients of q_i. By taking the product on each q_i, we obtain

$$\left(\prod q_i\right)(f) \in \bigcap_{i=1}^{s} \mathfrak{p}_i = \sqrt{0},$$

i.e. $(\prod q_i)(f)$ is nilpotent. So there is some $n \in \mathbb{N}$ such that $q = (q_1, \dots, q_s)^n$ and $q(f) = 0$. Moreover,

$$|f|_{\sup} = \max_{j=1,\dots,s} |f_j|_{\sup} = \max_{j=1,\dots,s} |\sigma(q_j)| \geqslant \sigma(q).$$

It deduces that $\sigma(q) = |f|_{\sup}$. □

Now we are ready to prove the maximum principle.

Proof of Theorem 1.13. As before we take minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of A and $f_i \in A/\mathfrak{p}_i$ residue classes. There is some i such that $|f|_{\sup} = |f_i|_{\sup}$. It reduces to replace A by $A/\mathfrak{p}_i$ for this fixed i. Thereby we assume A is an integral domain without loss of generality. Granting Lemma 1.16 we choose a finite monomorphism $T_d \hookrightarrow A$ and $p_f = \zeta^r + a_1\zeta^{r-1} + \cdots + a_r$, with $p_f(f) = 0$ and

$$|f|_{\sup} = \max_i |a_i|_{\sup}^{1/i}.$$

Note that the right-hand maximum lands in some $|a_{i_0}|_{\sup}^{1/i}$ for $1 \leqslant i_0 \leqslant s$ and $a_{i_0} \in T_d$. Via the maximum principle on T_d,

$$|a_{i_0}|_{\sup} = |a_{i_0}(x)|, \quad x \in \operatorname{Max} T_d,$$

where along $T_d \hookrightarrow A$, $x \in T_d$ admits inverse images $y_1, \dots, y_s \in A$. Also, by Lemma 1.16(2),

$$\max_i |f(y_i)| = |a_{i_0}(x)|^{1/i_0} = |a_{i_0}|_{\sup}^{1/i_0} = |f|_{\sup}$$

This shows the existence of y_i. □

Proposition 1.18. *Let A be an affinoid algebra with $f \in A$. Then there is an integer $n > 0$ such that*

$$|f|_{\sup}^n \in |K|.$$

This is basically because $|f|_{\sup} = |a_i|^{1/i}$ for some $a_i \in T_d$ by the maximum principle.

Theorem 1.19. *Let A be an affinoid algebra with $f \in A$. Fix $\alpha : T_n \twoheadrightarrow A$, and hence the extrinsic residue norm $|\cdot|_\alpha$. The following are equivalent.*

(1) $|f|_{\sup} \leqslant 1$;
(2) *There is some $a_1, \dots, a_r \in A$ such that*

$$f^r + a_1 f^{r-1} + \cdots + a_r = 0, \quad |a_i|_\alpha \leqslant 1;$$

(3) $\{|f^n|_\alpha\}_{n\in\mathbb{N}}$ *is a bounded sequence.*

Theorem 1.19 has the following respective version.

Theorem 1.20. *Resume the setups. The following are equivalent.*

(1) $|f|_{\sup} < 1$;
(2) $\{|f^n|_\alpha\}_{n\in\mathbb{N}}$ *is a zero sequence, i.e. $|f^n|_\alpha \to 0$ as $n \to \infty$.*

Our observation from the two theorems above can be relevant to the following.

Proposition 1.21. *Given any $\varphi : B \to A$ homomorphism of K-algebras, φ is continuous with respect to any residue norms on A and B. In particular, all residue norms are equivalent.*

Example 1.22. Let $A \in \mathcal{A}$. We define

$$A\langle \zeta_1, \dots, \zeta_n \rangle = \left\{ \sum_{\nu \in \mathbb{N}^n} a_\nu \zeta^\nu \in A[\![\zeta]\!] : \lim a_\nu = 0 \right\}.$$

There is a natural homomorphism

$$K\langle \zeta, \zeta_1, \dots, \zeta_n \rangle \longrightarrow A\langle \zeta_1, \dots, \zeta_n \rangle, \quad \zeta \longmapsto A,\ \zeta_i \longmapsto \zeta_i.$$

It can be checked that $A\langle \zeta_1, \dots, \zeta_n \rangle \in \mathcal{A}$ is an affinoid algebra.

Summary 1.23. Recall that for an affinoid K-algebra A with a choice $T_n \twoheadrightarrow A$, it is noetherian and Jacobson, and $T_d \hookrightarrow A$ is finite for some d. A morphism $A \to B$ in $\mathcal{A}$ corresponds to a morphism $\operatorname{Sp} A \to \operatorname{Sp} B$ in $\mathcal{B}$. Also, there are two types of norms on A:

- the residue norm $|\cdot|_\alpha : |\overline{f}|_\alpha = \inf_{\alpha(f)=\overline{f}} |f|$, which is a K-algebra norm;
- the supreme norm $|\cdot|_{\sup} : |\overline{f}|_{\sup} = \sup_{x \in \operatorname{Max} A} |\overline{f}(x)|$, which is an intrinsic semi-norm.

We have the following basic properties.

(1) By definition, we have $|\overline{f}|_{\sup} \leqslant |\overline{f}|_\alpha$ for any $f \in A$.
(2) Also, the maximum principle holds: For any $f \in A$ there is $x \in \operatorname{Max} A$ such that $|f(x)| = |f|_{\sup}$.
(3) Given $f \in A$, then $|f|_{\sup} \leqslant 1$ if and only if $\{|f^n|_\alpha\}$ is a bounded sequence; $|f|_{\sup} < 1$ if and only if $\{|f^n|_\alpha\}$ tends to be zero.[2]
(4) All residue norms are equivalent. Namely, all $\alpha : T_n \twoheadrightarrow A$ define equivalent topologies on A as induced quotient topologies of T_n.

2. Affinoid spaces

Similarly as in algebraic geometry, we will see the category of affinoid algebras is (oppositely) equivalent to the category of affinoid spaces. Recall that any element of the Tate algebra T_n can be regarded as the function $\mathbb{B}^n(\overline{K}) \to \overline{K}$. There is also a surjection $\mathbb{B}^n(\overline{K}) \to \operatorname{Max} T_n$.

Let $A = T_n/\mathfrak{a}$ be an affinoid algebra. Define

$$V(\mathfrak{a}) = \{x \in \mathbb{B}^n(\overline{K}) : f(x) = 0,\ \forall f \in \mathfrak{a}\}.$$

We expect that, in parallel to algebraic geometry, any element of A can be viewed as a function $V(\mathfrak{a}) \to \overline{K}$ as well.

Definition 2.1. For an affinoid algebra A, define its *affinoid K-space* as the pair of datum

$$\operatorname{Sp} A := (\operatorname{Max} A, A).$$

For any ideal $\mathfrak{a}$ of A, define the *vanishing locus of* $\mathfrak{a}$ by

$$\begin{aligned} V(\mathfrak{a}) &= \{x \in \operatorname{Sp} A : f(x) = 0,\ \forall f \in \mathfrak{a}\} \\ &= \{x \in \operatorname{Sp} A : \mathfrak{a} \subseteq x\}. \end{aligned}$$

[2] This is seen as a bridge between the intrinsic norm and the extrinsic norm. It indicates the independence of $|\cdot|_\alpha$ with respect to the choice of α.

Recall that in algebraic geometry we concern about prime ideals of a commutative ring instead of maximal ideals. We only work on $\operatorname{Max} A$ here because nice properties would arise by localizing A at some $\mathfrak{m} \in \operatorname{Max} A$ (see the upcoming section).

Proposition 2.2. *All ideals below are ideals of an affinoid algebra A.*

(1) *If $\mathfrak{a} \subseteq \mathfrak{b}$ then $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$.*
(2) $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.
(3) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

We observe that when $\mathfrak{a}$ runs through all ideals of A, $\{V(\mathfrak{a})\}$ is a family of closed subsets of $\operatorname{Sp} A$, as it admits any intersection and finite union. For convenience, we also name the Zariski topology that the topology defined by complements of $V(\mathfrak{a})$. Denote the basic open subsets

$$D_f := \{x \in \operatorname{Sp} A : f(x) \neq 0\}.$$

Proposition 2.3. *Those $\{D_f\}_{f \in A}$ form a basis of open subsets for the Zariski topology on $\operatorname{Sp} A$.*

Proposition 2.4 (Hilbert's Nullstellensatz)**.** *For any subset $Y \subseteq \operatorname{Sp} A$, define*

$$\begin{aligned}\operatorname{id}(Y) &:= \bigcap_{y \in Y} \mathfrak{m}_y \subseteq A \\ &= \{f \in A : f(y) = 0,\ \forall y \in Y\}.\end{aligned}$$

Then

$$V(\operatorname{id}(Y)) = \overline{Y}, \quad \operatorname{id}(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Corollary 2.5. *Let $\{f_i\} \in A$ be a family. The following are equivalent.*

- $\bigcup_i D_{f_i} = \operatorname{Sp} A$;
- $(f_1, \dots, f_n) = A$.

Given a homomorphism $\sigma : A \to B$, we have a natural morphism between formal spectrum spaces:

$$\operatorname{Sp} \sigma : \operatorname{Sp} B \longrightarrow \operatorname{Sp} A, \quad \mathfrak{m} \longmapsto \sigma^{-1}(\mathfrak{m}).$$

Recall that in algebraic geometry the inverse image of a maximal ideal generally fails to be maximal. Whereas in rigid geometry the same proposition is valid. Moreover,

$$K \hookrightarrow A/\sigma^{-1}(\mathfrak{m}) \hookrightarrow B/\mathfrak{m}.$$

Notation 2.6. Let $\mathcal{B}$ be the category of affinoid spaces. Its objects are all elements in form $\operatorname{Sp} A = (\operatorname{Max} A, A)$. And the morphisms of $\mathcal{B}$ are of form $\operatorname{Sp} A \to \operatorname{Sp} B$ induced by the "dual" homomorphism $B \to A$ in $\mathcal{A}$, the category of affinoid algebras.

It turns out that $\mathcal{B} \cong \mathcal{A}^{\mathrm{opp}}$. The two Cartesian diagrams correspond to each other:

$$\begin{array}{ccc} R & \longrightarrow & S_1 \\ \downarrow & & \downarrow \\ S_2 & \longrightarrow & S_1 \widehat{\otimes}_R S_2, \end{array} \qquad \begin{array}{ccc} \operatorname{Sp}(S_1 \widehat{\otimes}_R S_2) & \longrightarrow & \operatorname{Sp} S_1 \\ \downarrow & & \downarrow \\ \operatorname{Sp} S_2 & \longrightarrow & \operatorname{Sp} R. \end{array}$$

Summary 2.7. For an affinoid algebra A we define $\operatorname{Sp} A = (\operatorname{Max} A, A)$ as a topological space. Then

- The Zariski topology on $\operatorname{Sp} A$ is defined by closed subsets $V(\mathfrak{a}) = \{\mathfrak{m}_x \in \operatorname{Sp} A : \mathfrak{a} \subseteq \mathfrak{m}_x\}$ for all ideals $\mathfrak{a}$.
- The topology basis is given by principal open subsets $D_f = \{x \in \operatorname{Sp} A : f(x) \neq 0\}$.
- A morphism $\operatorname{Sp} A \to \operatorname{Sp} B$ is in correspondence with a morphism $B \to A$, which admits the functoriality, e.g. for fibre product, etc..

3. Affinoid subdomains

This section prepares for the definitions of rigid space together with sheaves on it, as well as the Grothendieck topology. In fact, we are to make affinoid subdomains the open subsets of Grothendieck topology on rigid spaces. The motivation to define affinoid subdomains is that the Zariski topology on $\operatorname{Sp} A$ is more than coarse so that we are to find another finer topology.

A priori we have

$$\mathbb{B}^n(\overline{K}) \twoheadrightarrow \operatorname{Max} T_n \approx \operatorname{Sp} T_n \supseteq \operatorname{Sp} A, \quad T_n \twoheadrightarrow A.$$

So the topology of $\operatorname{Sp} A$ can be viewed as the quotient topology induced from $\operatorname{Sp} T_n$, which further comes from $\mathbb{B}^n(\overline{K})$, the subset of an affine space $\mathbb{A}^n_{\overline{K}}$. Now for $X = \operatorname{Sp} A$ and $f \in A$, $\varepsilon \in \mathbb{R}_{>0}$, define

$$X(f; \varepsilon) := \{x \in X : |f(x)| \leqslant \varepsilon\}.$$

For simplicity we also denote

$$X(f) := X(f; 1), \quad X(f_1, \ldots, f_r) = X(f_1) \cap \cdots \cap X(f_r).$$

Definition 3.1. A *canonical topology* on $\operatorname{Sp} A$ is the topology generated by all $X(f; \varepsilon)$ for $f \in A$ and $\varepsilon \in \mathbb{R}_{>0}$, i.e. the coarsest topology under the finite intersection and arbitrary union of $X(f; \varepsilon)$'s.

Proposition 3.2. *For $X = \operatorname{Sp} A$, the canonical topology is generated by $X(f)$'s. Namely, a subset $U \subseteq X$ is open if and only if $U = \bigcup U_i$, where $U_i = X(f_{i_1}, \ldots, f_{i_{n(i)}})$.*

Proof. This is essentially because

$$X(f; \varepsilon) = \bigcup_{\substack{\varepsilon' \leqslant \varepsilon \\ \varepsilon' \in |\overline{K}^*|}} X(f; \varepsilon'),$$

for which we note that $|f(x)| \in |\overline{K}^*|$ for any $x \in K$. For each $\varepsilon' \in |\overline{K}^*|$, there is some $s \in \mathbb{N}$ such that $(\varepsilon')^s \in |\overline{K}^*|$ by the definition of valuation extension; i.e. there is some $c \in K^*$ such that $(\varepsilon')^s = |c|$. Hence

$$X(f; \varepsilon') = X(f^s; (\varepsilon')^s) = X(c^{-1} f^s; 1) = X(c^{-1} f^s).$$

□

Lemma 3.3. *Let $X = \operatorname{Sp} A$ and $f \in A$, $x \in \operatorname{Sp} A$, $\varepsilon = |f(x)| > 0$. Then there exists $g \in A$ such that $g(x) = 0$ and for any $y \in X(g)$ we have $|f(y)| = \varepsilon$. In particular, $X(g)$ is an open neighborhood of x contained in $\{y \in X : |f(y)| = \varepsilon\}$.*

Corollary 3.4. *Let* $\operatorname{Sp} A$ *be an affinoid* K*-space. Then, for* $f \in A$ *and* $\varepsilon \in \mathbb{R}_{>0}$*, the following sets are open with respect to the canonical topology:*

$$\begin{aligned}&\{x \in \operatorname{Sp} A : |f(x)| = \varepsilon\},\\&\{x \in \operatorname{Sp} A : f(x) \neq 0\},\\&\{x \in \operatorname{Sp} A : |f(x)| \leqslant \varepsilon\},\\&\{x \in \operatorname{Sp} A : |f(x)| \geqslant \varepsilon\}.\end{aligned}$$

Corollary 3.5. *Let* $X = \operatorname{Sp} A$ *be an affinoid* K*-space, and let* $x \in X$ *correspond to the maximal ideal* $\mathfrak{m}_x \subset A$*. Then the sets* $X(f_1, \ldots, f_r)$ *for* $f_1, \ldots, f_r \in \mathfrak{m}_x$ *and variable* r *form a basis of neighborhoods of* x*.*

Proof of Lemma 3.3. Consider the natural quotient map

$$A \longrightarrow A/\mathfrak{m}_x, \quad f \longmapsto \overline{f} = f(x).$$

Let $P(\zeta) = \zeta^n + c_1\zeta^{n-1} + \cdots + c_n \in K[\zeta]$ be the minimal polynomial of $f(x)$ over K. Also let

$$P(\zeta) = \prod_{i=1}^{n}(\zeta - \alpha_i), \quad \alpha_i \in \overline{K}$$

be the root factorization on $\overline{K}$. Then for each i,

$$\varepsilon = |f(x)| = |\overline{f}| = |\alpha_i|$$

by definition of $|\cdot|$ on $A/\mathfrak{m}_x$. Let $g = P(f) \in A$. Then $g(x) = P(f(x)) = 0$.

Claim. For each $y \in X$, $|g(y)| < \varepsilon^n$ implies $|f(y)| = \varepsilon$.

If not, $|f(y) - \alpha_i| = \max(|f(y)|, |\alpha_i|) \geqslant \varepsilon$, but

$$|g(y)| = |P(f(y))| = \prod |f(y) - \alpha_i| \geqslant \varepsilon^n,$$

which leads to a contradiction. Hence $|f(y)| = \varepsilon$. Finally, one can choose $c \in K^*$ with $|c| < \varepsilon^n$, so that

$$X(c^{-1}g) \subseteq \{y : |f(y)| = \varepsilon\}.$$

This completes the proof by replacing g with $c^{-1}g$ if necessary. □

Proposition 3.6. *Let* $(\varphi, \varphi^*) : \operatorname{Sp} B \to \operatorname{Sp} A$ *be a morphism between affinoid spaces, where* $\varphi^* : A \to B$ *is the corresponding morphism of affinoid algebras. Then for* $f_1, \ldots, f_r \in A$*, we have*

$$\varphi^{-1}((\operatorname{Sp} A)(f_1, \ldots, f_r)) = \operatorname{Sp}(B)(\varphi^*(f_1), \ldots, \varphi^*(f_r)).$$

In particular, φ^{-1} *pullbacks an open set of canonical topology to an open set. So any morphism of affinoid spaces is continuous with respect to the canonical topology.*

Proof. For each $y \in \operatorname{Sp} B$ corresponding to $\mathfrak{m}_y \in \operatorname{Max} B$, the diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi^*\ } & B \\ \downarrow & & \downarrow \\ A/\mathfrak{m}_{\varphi(y)} & \longrightarrow & B/\mathfrak{m}_y \end{array}$$

with a monomorphism in the lower row. As we may embed the latter into $\overline{K}$, we see that $|f(\varphi(y))| = |\varphi^*(f)(y)|$ holds for any $f \in A$. This implies

$$\varphi^{-1}((\operatorname{Sp} A)(f)) = (\operatorname{Sp} B)(\varphi^*(f))$$

and, hence, forming intersections, we are done. □

Definition 3.7. Let $X = \operatorname{Sp} A$ be an affinoid K-space.

(1) A subset in X of type

$$X(f_1, \dots, f_r) = \{x \in X : |f_i(x)| \leqslant 1\}$$

for functions $f_1, \dots, f_r \in A$ is called a *Weierstrass domain* in X.

(2) A subset in X of type

$$X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{x \in X : |f_i(x)| \leqslant 1, |g_j(x)| \geqslant 1\}$$

for functions $f_1, \dots, f_r, g_1, \dots, g_s \in A$ is called *a Laurent domain* in X.

(3) A subset in X of type

$$X\left(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0}\right) = \{x \in X : |f_i(x)| \leqslant |f_0(x)|\}$$

for functions $f_0, \dots, f_r \in A$ without common zeros is called a *rational domain* in X.

Lemma 3.8. *All these three types of domains are open with respect to canonical topology.*

Proof. The openness of Weierstrass and Laurent domains can be read from the assertion of Lemma 3.3. In the case of a rational domain the same is true, as for any $x \in X\left(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0}\right)$ with $|f_0(x)| \neq 0$, it has an open neighborhood

$$X(f_1; f_1(x)) \cap \cdots \cap X(f_r; f_r(x)) \cap \{y : |f_0(y)| \geqslant |f_0(x)|\}$$

as a finite intersection of open subsets. □

The following example shows that we cannot drop the condition that $(f_0, \dots, f_r) = A$ in Definition 3.7(3).

Example 3.9. Let $X = \operatorname{Sp} T_1 = \operatorname{Sp} K\langle\zeta\rangle$,[3] $c \in K$ with $0 < |c| < 1$. Then

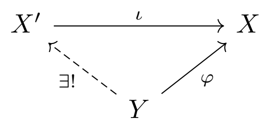
$$\begin{aligned} X\left(\frac{\zeta}{c\zeta}\right) &= \{x \in X : |\zeta(x)| \leqslant |c\zeta(x)|\} \\ &= \{x \in X : |\zeta(x)| = 0\} \\ &= \{\mathfrak{m}_{(\zeta)}\}. \end{aligned}$$

It turns out that this single point cannot be an open subset. If $\mathfrak{m}_{(\zeta)} \in X$ is open, then there are $f_1, \dots, f_r \in \mathfrak{m}_{(\zeta)}$ such that $\{\mathfrak{m}_{(\zeta)}\} = X(f_1, \dots, f_r)$. On the other hand, one can also find $d \in K$ with $|d| < 1$ such that $\zeta - d$ is a prime element. (If $|d| \geqslant 1$ then $\zeta - d$ is possibly a unit.) Whenever $|d|$ is sufficiently small, we have $|f_i(\mathfrak{m}_{(\zeta-d)})| < 1$ for any i. Thus $\mathfrak{m}_{(\zeta)} \neq \mathfrak{m}_{(\zeta-d)} \in X(f_1, \dots, f_r)$. This leads to a contradiction. Therefore, the rational domain $\mathfrak{m}_{(\zeta)}$ is not open.

Here comes more general affinoid subdomains.

[3]Recall that $K\langle\zeta\rangle$ is a Euclidean domain.

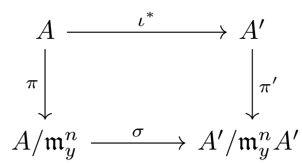
Definition 3.10. Let $X = \operatorname{Sp} A$ be an affinoid K-space. A subset $U \subseteq A$ is called an *affinoid subdomain of* X if there exists an affinoid K-space X' together with a morphism $\iota : X' \to X$ such that $\operatorname{im} \iota \subseteq U$, and for any morphism $\varphi : Y \to X$ such that $\varphi(Y) \subseteq U$, φ factors uniquely through ι, i.e.



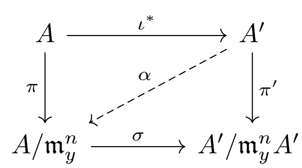
Lemma 3.11. *Let $X = \operatorname{Sp} A$, $X' = \operatorname{Sp} A'$ with $U \subseteq X$ an affinoid subdomain. Let $(\iota, \iota^*) : X' \to X$ and $\iota^* : A \to A'$ be the datum of the K-morphism by definition. Then*

(1) *ι is injective as a map, and $\iota(X') = U$. Hence it induces a bijection $X' \xrightarrow{\sim} U$.*
(2) *For any $x \in X'$ and $n \in \mathbb{N}$, the map ι^* induces an isomorphism of affinoid K-algebras $A/\mathfrak{m}_{\iota(x)}^n \xrightarrow{\sim} A'/\mathfrak{m}_x^n$.*
(3) *For $x \in X'$ we have $\mathfrak{m}_x = \mathfrak{m}_{\iota(x)} A'$.*

Proof. Choose $y \in U$ that corresponds to $\mathfrak{m}_y \in \operatorname{Max} A$. Then



Note that $\mathfrak{m}_y$ is the unique maximal ideal in $A/\mathfrak{m}_y^n$, we see $\operatorname{Sp} A/\mathfrak{m}_y^n$ is a single point space; and along the morphism $\operatorname{Sp} A/\mathfrak{m}_y^n \to \operatorname{Sp} A$ the image is a single point as well. On the other hand, by the universal property of $\iota : \operatorname{Sp} A' \to \operatorname{Sp} A$, we see $\operatorname{Sp} A/\mathfrak{m}_y^n \to \operatorname{Sp} A$ factors through a unique morphism $\operatorname{Sp} A/\mathfrak{m}_y^n \to \operatorname{Sp} A'$. Hence there is $\alpha : A' \to A/\mathfrak{m}_y^n$ such that both the upper and the lower triangles are commutative:



Note that $\pi' = \sigma \circ \alpha$ is surjective, and hence σ is surjective; also, $\pi = \alpha \circ \iota^*$ is surjective, and so also is α. Note also that $\ker \pi' = \mathfrak{m}_y^n A' \subseteq \ker \alpha$. For any $s \in \ker \sigma$ there is $t \in A'$ such that $\alpha(t) = s$ since α is surjective. Then $\pi'(t) = \sigma(\alpha(t)) = \sigma(s) = 0$. So $t \in \ker \pi' \subseteq \ker \alpha$, and then $\alpha(t) = 0$, $s = \alpha(t) = 0$. This shows that σ is injective.

Now for $n = 1$, we have $\mathfrak{m}_{\iota(x)} A' = \mathfrak{m}_x$. This proves (1)(3). Then we get (2) from the bijectivity of σ and from the fact that $\mathfrak{m}_x = \mathfrak{m}_y A' = \mathfrak{m}_{\iota(x)} A'$. ☐

Proposition 3.12. *For any affinoid K-space $X = \operatorname{Sp} A$, Weierstrass, Laurent, and rational domains in X are examples of open affinoid subdomains. These are called special affinoid subdomains.*

Proof. Before proving this main result, we need a sublemma for the sake of checking the condition $\operatorname{im} \iota \subseteq U$, where U is the candidate affinoid subdomain. Given $(\varphi, \varphi^*) : Y \to X$ a morphism of affinoid K-spaces and $f_1, \dots, f_r \in A$, for any $y \in Y$, we have seen in Proposition 3.6 that

$$|\varphi^*(f_i)(y)| = |f_i(\varphi(y))|.$$

Then

$$\begin{aligned} \varphi(Y) \subseteq X(f_1, \dots, f_r) &\iff |f_i(\varphi(y))| \leqslant 1 \text{ for all } i, \\ &\iff |\varphi^*(f_i)(y)| \leqslant 1 \text{ for all } i, \\ &\iff |\varphi^*(f_i)|_{\sup} \leqslant 1 \text{ for all } i. \end{aligned}$$

Here the last equivalence is due to the maximum principle.

(a) *Weierstrass domain.* Denote $f = (f_1, \dots, f_r)$ and consider $X(f) \subseteq X$. Let

$$A\langle f \rangle = A\langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_1 - f_1, \dots, \zeta_r - f_r).$$

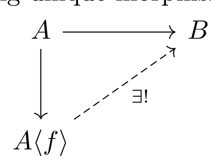
Then we have a natural morphism $\iota^* : A \to A\langle f \rangle$ corresponding to $\iota : \operatorname{Sp} A\langle f \rangle \to \operatorname{Sp} A = X$. We assert that this is the desired morphism in Definition 3.10.

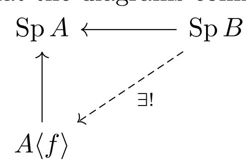
- We first check that $\operatorname{im} \iota \subseteq X(f)$. For $\iota^* : A \to A\langle f \rangle$,

$$|\iota^*(f_i)|_{\sup} = |\zeta_i|_{\sup} \leqslant |\zeta_i|_\alpha \leqslant 1.$$

Hence $\operatorname{im} \iota \subseteq X(f)$ follows from the sublemma.

- We then tackle with the universal property. Assume $\varphi : Y = \operatorname{Sp} B \to X = \operatorname{Sp} A$ is such that $\varphi(Y) \subset X(f)$, or equivalently $|\varphi^*(f_i)|_{\sup}| \leqslant 1$ for each i. It suffices to find the following unique morphisms such that the diagrams commute.

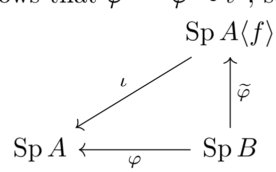




For this, we can extend $\varphi^* : A \to B$ to a morphism

$$\widetilde{\varphi}^* : A\langle \zeta_1, \dots, \zeta_r \rangle \longrightarrow B, \quad \zeta_i \longmapsto \varphi^*(f_i).$$

Then $\zeta_i - f_i \in \ker \widetilde{\varphi}^*$. It follows that $\varphi^* = \widetilde{\varphi}^* \circ \iota^*$, so



is a commutative diagram.

(b) *Laurent domain.* The argument is similar by constructing

$$A\langle f, g^{-1} \rangle = A\langle \zeta_1, \dots, \zeta_r, \xi_1, \dots, \xi_r \rangle / (\zeta_i - f_i, \xi_j g_j - 1)_{i,j}.$$

Then one can check the condition $\operatorname{im} \iota \subseteq X(f, g^{-1})$ for $\iota : \operatorname{Sp} A\langle f, g^{-1} \rangle \to X = \operatorname{Sp} A$ by noticing that

$$\zeta_i - \iota^* f_i = 0, \quad |\zeta_i|_{\sup} \leqslant 1,$$

and

$$\iota^*(g_j)\xi_j = 1, \quad |\xi_j|_{\sup} \leqslant 1.$$

The universal property would be verified in a similar way.

(c) *Rational domain.* The details are different from (a)(b), but the idea is the same. Construct

$$A\left\langle \frac{f}{f_0} \right\rangle = A\langle \zeta_1, \ldots, \zeta_r\rangle / (f_1 - f_0\zeta_1, \ldots, f_r - f_0\zeta_r)$$

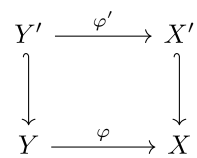
to finish the proof.

□

Proposition 3.13 (Transitivity of affinoid subdomains)**.** *Let $U \subset V$ and $V \subset X = \operatorname{Sp} A$ be affinoid subdomains. Then $U \subset X$ is an affinoid subdomain as well.*

Proposition 3.14. *Let $\varphi : Y \to X$ be a morphism of affinoid spaces, and $X' \subset X$ an affinoid subdomain. Then*

(1) *There is a unique $\varphi' : Y' \to X'$ such that the diagram*



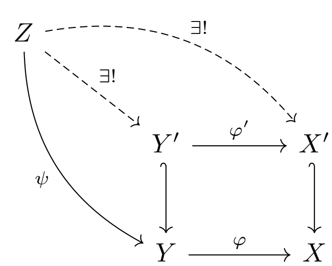
is Cartesian, i.e. $Y' = \varphi^{-1}(X') = Y \times_X X'$.

(2) *We have*

$$\varphi^{-1}(X(f)) = Y(\varphi^* f),$$
$$\varphi^{-1}(X(f, g^{-1})) = Y(\varphi^* f, (\varphi^* g)^{-1}),$$
$$\varphi^{-1}\left(X\left(\frac{f}{f_0}\right)\right) = Y\left(\frac{\varphi^* f}{\varphi^* f_0}\right), \quad (f, f_0) = A.$$

Namely, morphisms of affinoid spaces preserve special affinoid subdomains.

Proof. Note that (2) is implied by (1) because, for example, $Y \times_X X(f) = \varphi^{-1}(X(f)) = Y(\varphi^* f)$ once (1) is valid. We take $\psi : Z \to Y$ such that $\operatorname{im}\psi \subseteq Y'$ and then $\operatorname{im}(\varphi \circ \psi) \subset \varphi(Y') \subset X'$. Hence there exists a unique morphism $Z \to X'$ that $\psi \circ \varphi$ factors through, by the universal property.



So we can take $Y' = \varphi^{-1}(X')$, and there consequently exists a unique $Z \to Y'$ such that $\psi : Z \to Y' \to Y$. Hence $Y' \simeq Y \times_X X'$ is an affinoid subdomain of Y. □

Proposition 3.15. *Let $U, V \subseteq X$ be general (resp. Weierstrass/Laurent/rational) affinoid subdomains. Then so also is $U \cap V$. In particular,*

$$\begin{Bmatrix} \textit{Weierstrass domains} \\ X(f) \end{Bmatrix} \subset \begin{Bmatrix} \textit{Laurent domains} \\ X(f, g^{-1}) \end{Bmatrix} \subset \begin{Bmatrix} \textit{rational domains} \\ X(f/f_0) \end{Bmatrix}.$$

Proof. We apply Proposition 3.14 to see the following is a Cartesian diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ U & \longrightarrow & X \end{array}$$

so that $U \cap V$ is an affinoid subdomain of U. Suppose $U = X(f/f_0)$ and $V = X(g/g_0)$ are rational subdomains. We claim that

$$U \cap V = X\left(\frac{f_i g_j}{f_0 g_0}\right)_{\substack{0<i\leqslant r \\ 0<j\leqslant s}}.$$

To check this, we first note that $(f, f_0) = (g, g_0) = A$ implies $(f_i g_j)_{i,j\geqslant 0} = A$. And for each $x \in U \cap V$,

$$|(f_i g_j)(x)| \leqslant |(f_0 g_0)(x)| \implies |(f_i g_0)(x)| \leqslant |(f_0 g_0)(x)| \implies |f_i(x)| \leqslant |f_0(x)|.$$

This completes the proof. □

We finish this section with the classification theorem of affinoid subdomains.

Theorem 3.16 (Gerritzen-Grauert)**.** *Let $U \subseteq X = \operatorname{Sp} A$ be an affinoid subdomain. Then*

$$U = \bigcup_{i=1}^{n} X(f_i/f_{0,i}), \quad f_i = (f_{i_1}, \ldots, f_{i_{k(i)}}).$$

Namely, any general affinoid subdomain is a finite union of some rational subdomains. In particular, any affinoid subdomain is open with respect to canonical topology.

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