§ 3.1 Affinoid Algebras.

Def 1: A K-alg A is called an affinoid algebra if there exists an epic morphism of K-algebras $\alpha: T_n \twoheadrightarrow A$ for some $n \in \mathbb{N}$.

Prop 2. (The category of affinoid admits (complete) tensor product)

Prop 3. A is an affinoid K-alg.

(revisited. § 2.2).

(i). A is Noetherian

(ii) A is Jacobson

(iii). Noether Normalization

$\exists\, d \in \mathbb{Z}_{\geq 0}$ $T_d \hookrightarrow A$. finite.

Prop 4. A is affinoid K-alg.

$\mathfrak{q} \subset A$ s.t. $\sqrt{\mathfrak{q}} = \mathfrak{m}$ is maximal.

$\Rightarrow$ $A/\mathfrak{q}$ is finite dimensional over K.

Pf: $A/\mathfrak{q}$ is finite dimensional

$\Leftarrow A/\mathfrak{m}^k$ is ________

$$A \rightsquigarrow A_{\mathfrak{m}}.$$

$$\underline{A_{\mathfrak{m}}/\mathfrak{m}^k A_{\mathfrak{m}}} = A/\mathfrak{m}^k.$$

$$A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}, \quad \underline{\mathfrak{m}^i A_{\mathfrak{m}}/\mathfrak{m}^{i+1} A_{\mathfrak{m}}}.$$

$A/\mathfrak{m}$: finite dimensional

$$\dim_K A/\mathfrak{m} < \infty.$$

Def : Residue norm :

A : affinoid K-alg.

$\alpha : T_n \twoheadrightarrow A$.

$|\cdot|_\alpha : A \to \mathbb{R}_{\geq 0}$.

$f \in T_n$. $\underline{\alpha(f)} \mapsto \inf_{a \in \ker\alpha} |f-a|$.

Proposition 5. *For an ideal $\mathfrak{a} \subset T_n$, view the quotient $A = T_n/\mathfrak{a}$ as an affinoid K-algebra via the projection map $\alpha : T_n \longrightarrow T_n/\mathfrak{a}$. The map $|\cdot|_\alpha : T_n/\mathfrak{a} \longrightarrow \mathbb{R}_{\geq 0}$ satisfies the following conditions:*

(i) *$|\cdot|_\alpha$ is a K-algebra norm, i.e. a ring norm and a K-vector space norm, and it induces the quotient topology of T_n on $T_n/\mathfrak{a}$. Furthermore, $\alpha : T_n \longrightarrow T_n/\mathfrak{a}$ is continuous and open.*

(ii) *$T_n/\mathfrak{a}$ is complete under $|\cdot|_\alpha$.*

(iii) *For any $\overline{f} \in T_n/\mathfrak{a}$, there is an inverse image $f \in T_n$ such that $|\overline{f}|_\alpha = |f|$. In particular, for any $\overline{f} \in T_n/\mathfrak{a}$, there is an element $c \in K$ with $|\overline{f}|_\alpha = |c|$.*

Pf : (i)

$|\overline{f}|_\alpha = 0 \iff \overline{f} = 0 \iff f \in \ker\alpha$.

($\mathfrak{a}$ is closed).

$\underline{|\overline{f}\,\overline{g}| \leq |\overline{f}| \cdot |\overline{g}|}$.

$$|\bar{f}|_\alpha = \inf |\quad|.$$
$$= |f-a| \quad \text{for some } a \in \mathfrak{a}.$$

$$|\bar{g}|_\alpha = |g-b|. \quad b \in \mathfrak{a}$$

$$|\bar{f}\bar{g}|_\alpha \le |(f-a)(g-b)|$$
$$= |(f-a)|\cdot|g-b|$$
$$= |\bar{f}|_\alpha \cdot |\bar{g}|_\alpha$$

$$|\bar{f}+\bar{g}|_\alpha \le \max\{|\bar{f}|_\alpha, |\bar{g}|_\alpha\}.$$

$$|c\bar{f}|_\alpha = |c|\cdot|\bar{f}|. \quad \text{clear.}$$

Openness of α:

$$\alpha(B(0,\varepsilon)) = B_\alpha(0,\varepsilon).$$

$\subset$: Simple

$\supset$: θ is strictly closed.

Continuity:

$$\underline{|\alpha(f)|_\alpha \le |f|.}$$

(ii) T_n/θ is complete under $|\cdot|_\alpha$.

Sketch:

$$\begin{array}{ccc} T_n & \quad & x_1 \ \cdots \ x_n \quad \text{Cauchy?} \\ \Downarrow & & \uparrow \\ T_n/\theta. & & y_1, \cdots, y_n, \cdots \end{array}$$

Cauchy

$$|x_{n+1} - x_n| = |y_{n+1} - y_n|_\alpha.$$

(iii): $|f|_\alpha = |f|_{T_n} = |c|$ for some $c \in K$.

$\swarrow$ def.

$|K|$.

$\alpha: T_n \twoheadrightarrow A$. $\ker\alpha = \mathfrak{a} \subseteq T_n$

① $|\cdot|_\alpha$.

② $f: V(\mathfrak{a}) \subset B^n(\overline{K})$.

$\uparrow$

$A \longrightarrow \overline{K}$.

$|f|_{sup} = \sup_{x \in \operatorname{Max} A} |f(x)|$.

$f(x) \in A/x \hookrightarrow \overline{K}$.

finite over K.

$|f|_{\sup} = 0 \iff f \in A$ is nilpotent.

<u>Prop 6</u>: Power multiplicative:

$$|f^n|_{\sup} = |f|_{\sup}^n, \quad \forall f \in A, \quad \forall n \in \mathbb{Z}_{\geq 1}.$$

<u>Prop 7</u>: $\varphi : B \to A$ is a morphism between affinoid K-algebras.

Then $|\varphi(b)|_{\sup} \leq |b|_{\sup}, \ \forall b \in B$.

<u>Pf</u>: $\underset{m \in \operatorname{Max} A}{|\varphi(b)(m)|} \ \underline{\leq} \ \underset{\exists\, n \in \operatorname{Max} B}{|b(n)|}$.

Choose $n = \varphi^{-1}(m) \subset B$.

n is maximal.

finite ext over K

$K \hookrightarrow \underline{B/n} \hookrightarrow A/m$.

is a field

$|\varphi(b)(m)| = |b(m)|$. $\forall m \in \max A$

$\Longrightarrow |\varphi(b)|_{sup} \leq |b|_{sup}$. $\boxed{/\!/}$

<u>Prop 8</u>: $|\cdot|_{sup} = |\cdot|$ (Gauss norm) on a Tate algebra T_n.

Pf: $f \in T_n$.

$$|f| = \max\{f(x) : x \in B^n(\overline{K})\}$$

(maximum principle)

$$= |f|_{sup}.$$

Recall: $B^n(\overline{K}) \twoheadrightarrow \max T_n$,

$x = (x_1, \cdots, x_n) \longmapsto m_x$

$= \ker(T_n \to \overline{K})$

$f \mapsto f(x_1, \cdots, x_n)$

Prop 9. A : affinoid K-alg.

$\alpha : T_n \twoheadrightarrow A$.

$|\cdot|_\alpha$: Residue norm.

Then $|f|_{\sup} \leq |f|_\alpha, \ \forall f \in A$.

$\Rightarrow |f|_{\sup}$ is finite.

Pf: Choose $m \in \max A$. and $f \in A$.

$$\begin{array}{ccc} m & \in & \max A \\ | & & | \\ n = \alpha^{-1}(m) & \in & \max T_n \end{array}$$

$f \in A \xrightarrow{\text{preimg}} g \in T_n$. $|g| = |f|_\alpha$.

$|f(m)| = |g(n)| \leq |g| = |f|_\alpha$.

☒

Prop 10. $|f|_{\sup} = 0$

$\Updownarrow$ [1].

f is nilpotent.

Pf: A is Jacobson. [L6].

- $|\cdot|_{\sup}$ & $|\cdot|_\alpha$.

Lem 11.

$$p(\xi) = \xi^r + c_1 \xi^{r-1} + \dots + c_r \in K[\xi]$$

$$= \prod_{j=1}^{r} (\xi - \alpha_j), \quad \alpha_j \in \overline{K}.$$

Then $\max_{j=1,\dots,r} |\alpha_j| = \max_{i=1,\dots,r} |c_i|^{\frac{1}{i}}$.

Pf: $\geq$:

$$c_i = \pm \sum_{n_1 < \dots < n_i} \alpha_{n_1} \cdots \alpha_{n_i}$$

$$|c_i| \leq \max \{ |\alpha_{n_1}| \cdots |\alpha_{n_i}| \}.$$

$$\leq \left(\max_{\delta=1,\cdots,r} |\alpha_\delta|\right)^i$$

$\leq$:

$I \subset \{1, \cdots, r\}$. s.t.

$$|\alpha_i| = \max_{\delta=1,\cdots,r} |\alpha_\delta| \iff i \in I.$$

$t = |I|$

$$|c_t| = \left|\sum_{i_1,\cdots,i_t} \alpha_{i_1} \cdots \alpha_{i_t}\right|.$$

$$= \left|\prod_{i \in I} \alpha_i\right| = \left(\max_{\delta=1,\cdots,r} |\alpha_\delta|\right)^t.$$

$$\Rightarrow |c_t|^{\frac{1}{t}} = \max |\alpha_\delta|. \qquad \boxed{\checkmark}.$$

$\sigma(p)$. $p \in A[\xi]$. A : normed ring.

$$\sigma(p) = \max_{i=1,\dots,r} |c_i|^{\frac{1}{i}} \quad \text{(Spectral value)}$$

where $p = \xi^r + c_1\xi^{r-1} + \cdots + c_r$.

In lem 11 : $\max|\alpha_i| = \sigma(p)$.

<u>Lem 12</u>: A is a (semi-)normed ring.

$p, q \in A[\xi]$. are monic.

$$\sigma(pq) \le \max(\sigma(p), \sigma(q)).$$

<u>Pf</u>: $p = \sum_{i=0}^{m} a_i \xi^{m-i}$ $\quad a_0 = b_0 = 1$.

$$q = \sum_{j=0}^{n} b_j \xi^{n-j}$$

$$pq = \sum_{\lambda=0}^{m+n} c_\lambda \xi^{m+n-\lambda} \quad c_\lambda = \sum_{i+j=\lambda} a_i b_j.$$

$$|c_\lambda| \le \max(\sigma(p), \sigma(q))^\lambda.$$

We know

$$|a_i| \le \sigma(p)^i \le \max(\sigma(p), \sigma(q))^i$$

$$|b_j| \le \sigma(q)^j \le \underline{\qquad\qquad\qquad}^{\,j}.$$

$$|c_\lambda| = \left|\sum_{i+j=\lambda} a_i b_j\right|$$

$$\le \max_{i+j=\lambda} |a_i|\,|b_j|$$

$$\le \max_{i+j=\lambda} \left(\max(\sigma(p), \sigma(q))\right)^{\overbrace{i+j}}.$$

(I).

Lem 13. $T_d \hookrightarrow A$: finite monic of K-alg.

A is torsion-free as T_d-mod

$f \in A$.

(i). $\exists!$ monic $P_f = \xi^n + a_1 \xi^{n-1} + \cdots + a_n$ $\in T_d[\xi]$ of minimal deg s.t $P_f(f) = 0$.

And $\ker \begin{bmatrix} T_d[\xi] \longrightarrow A \\ \xi \longmapsto f \end{bmatrix}$ is generated by P_f as a T_d-hom.

(ii).
$$\begin{tikzcd} y_1, \dots, y_s \arrow[r, phantom, "\in"] \arrow[d, dash] & A \arrow[d, dash] \\ x \arrow[r, dash] & T_d \end{tikzcd}$$
$y_1,\dots,y_s$: max A, x : max T_d.

Then
$$\max_{j=1,\dots,s} |f(y_j)| = \max_{i=1,\dots,n} |a_i(x)|^{\frac{1}{i}}$$

(iii) ($\Leftarrow$ (ii)) by taking sup over Max T_d)

$$|f|_{\sup} = \max_{i=1,\dots,r} |a_i|_{\sup}^{\frac{1}{i}}$$

Pf: $F = Q(T_d)$. $F(A) = A \otimes_{T_d} F$

$\exists$ diag of inclusions

$$\begin{tikzcd} T_d \arrow[r, hook] \arrow[d] & A \arrow[d, hook] \\ F \arrow[r, hook] & F(A) \end{tikzcd}$$

Torsion free.

induce

localization

$$\ker\left(\begin{array}{ccc} F[\xi] & \twoheadrightarrow & F(A) \\ \xi & \mapsto & f \end{array} \right) = (P_f).$$

$P_f \in F[\xi]$.

Claim: $p_f \in T_d[s]$.

$T_d \hookrightarrow A$ finite.

$f \in A \Rightarrow h \in T_d[s]$ $(\subseteq F[s])$.

$h(f) = 0$. h is monic.

$\Rightarrow p_f \mid h$ over $F[s]$.

$\Rightarrow p_f \mid h$ over $T_d[s]$, as T_d is factorial.

$(h \in T_d[s],\ h(f) = 0 \Rightarrow p_f \mid h)$.

Thus $T_d[s] \longrightarrow A$ has kernel

$s \longmapsto f$.

(p_f).

(ii)

$$\begin{tikzcd}
 & y_1 \cdots y_\xi \arrow[r, dash] \arrow[d, dash] & A \arrow[d, dash] \\
y_\xi \arrow[ur, dash] & X \arrow[r, dash] & T_d
\end{tikzcd}
\qquad
\begin{tikzcd}
f \in A \\
f \in T_d[f] \arrow[u] \\
T_d \arrow[u]
\end{tikzcd}$$

$$\operatorname{Max} A \twoheadrightarrow \operatorname{Max} T_d[f] \twoheadrightarrow \operatorname{Max} T_d$$

$$y_\xi \text{ —— } z_i \text{ —— } X$$

$$|f(y_\xi)| = |f(z_i)|$$

$$T_d/X \hookrightarrow \underset{\bar{f}}{T_d[f]/z_i} \hookrightarrow \underset{\bar{f}}{A/y_\xi}$$

$$\max|f(y_\xi)| = \max|f(z_i)|$$

Thus WLOG. $A = T_d[f]$

$$A = T_d[f] \overset{\text{by (i)}}{=} T_d[y]/(P_f).$$

$$\begin{tikzcd}
y_1, \dots, y_s \arrow[r, dash] \arrow[d, dash] & T_d[f] = T_d[y]/(P_f) \arrow[d, dash] \arrow[dr, phantom, "\shortparallel"] & \\
X \arrow[r, dash] & T_d & A
\end{tikzcd}$$

Set $L = T_d/X$ (is finite over k)

$$A/X = L[y]/(\overline{P_f}).$$

$\overline{P_f}$ has roots in $\bar{L}$, denoted by $\alpha_1, \dots, \alpha_r$

$$\underline{\max_j |f(y_j)|} = \max_{i=1,\dots,r} |\alpha_i|.$$

$$\Big\Updownarrow \; \overline{P_f}.$$

$$= \max_i |a_i(X)|^{\frac{1}{i}} \qquad \square$$

<u>Lem 14</u>. Let $\varphi : B \to A$ is finite hom between affinoid K-algebras.

Then for any $f \in A$. $\exists$ equation.

$$f^n + b_1 f^{n-1} + \cdots + b_n = 0. \quad b_i \in B.$$

$$|f|_{\sup} = \max_{i=1,\dots,n} |b_i|_{\sup}^{\frac{1}{i}}$$

Thm 15. A is an affinoid K-algebra. $f \in A$.

Then $\exists\, x \in \operatorname{Max} A$ s.t.

$$|f(x)| = |f|_{\sup}.$$

Pf: A. minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$.

WLOG. A is an int domain.

$\Rightarrow$ A finite monic $T_d \hookrightarrow A$.

Choose $f \in A$.

$$f^r + a_1 f^{r-1} + \cdots + a_r = 0 \quad \text{over } T_d.$$

$$\Rightarrow \max_{j=1,\ldots,s} |f(y_j)| = \max_{i=1,\ldots,r} |a_i(x)|^{\frac{1}{i}}$$

where

$$\begin{tikzcd}
y_j \arrow[r, dash] \arrow[d, dash] & A \\
x \arrow[r, dash] & T_d \arrow[u, dashed]
\end{tikzcd}$$

$$|a_1 \cdots a_r(x)| = |a_1 \cdots a_r|$$

$$\begin{array}{ccc} \| & & \| \\ \prod |a_i(x)| & & |a_1| \cdots |a_r| \end{array}$$

for some $x \in \max T_d$.

$$\Rightarrow \quad |a_i(x)| = |a_i|, \ \forall i.$$

$$\max_{y_\xi} |f(y_\xi)| = \max_{i=1,\dots,r} |a_i(x)|^{\frac{1}{i}}$$

$$= \max_{i=1,\dots,r} |a_i|^{\frac{1}{i}}$$

$$= |f|_{\sup}. \quad (\text{Lem } (4)).$$

$\square$.

Prop 16. A : Affinoid K-alg

$f \in A$.

$\Rightarrow \exists n \in \mathbb{Z}_{\geq 0}$ s.t. $|f|_{\sup}^{n} \in |K|$

Pf: Lem 14: $T_d \xrightarrow{\text{finite}} A$.

$$|f|_{\sup} = \underbrace{\max_{i=1,\dots,r} |b_i|_{\sup}^{\frac{1}{i}}}$$

$$= |b_\delta|_{\sup}^{\frac{1}{\delta}}.$$

$$|f|_{\sup}^{\delta} = |b_\delta|_{\sup} \in |K|.$$

Thm 17. $f \in A$. TFAE.

(i) $|f|_{\sup} \le 1$.

(ii) $\exists\, f^n + a_1 f^{n-1} + \cdots + a_n = 0$ with coefficient $a_i \in A$ s.t. $|a_i|_\alpha \le 1$.

(iii). $|f^n|_\alpha$ is bounded.

Sketch: $T_d \xhookrightarrow{\text{finite}} A$

(i) $\Rightarrow$ (ii) By lem 14:

$\exists$ equation with $|a_i|_{\sup} = |a_i| \le 1$.

where $a_i \in T_d$.

$$T_d \hookrightarrow T_n \overset{\alpha}{\twoheadrightarrow} A.$$

Contrative $\Rightarrow |a_i|_\alpha \le 1$.

(ii) $\Rightarrow$ (iii).

$$A^\circ = \{g \in A : |g|_a \le 1\}.$$

(ii) $\Rightarrow$ $A^\circ[f]$ is finite over A°.

norms are bounded

$\Rightarrow$ $|f^n|_a$ is bounded.

(iii) $\Rightarrow$ (i). $|f|_{sup}^n = |f|_{sup}^n \le |f^n|_a$. $\forall n \ge 1$

$\Rightarrow |f|_{sup} \le 1$.

Cor 18. TFAE. $\alpha : T_n \twoheadrightarrow A$.

(i). $|f|_{sup} < 1$.

(ii). $|f^n|_\alpha$ is a zero seq.

Sketch: (i) $\Rightarrow$ (ii).

$$|f^r|_{sup} = |c|. \quad c \in K.$$

$$\Rightarrow |c^{-1} f^r|_{sup} = 1.$$

Thm 17. $\Rightarrow$ $\{|c^{-n} f^{rn}|_\alpha\}_n$ is bounded.

$\Rightarrow \{|f^{rn}|_\alpha\}_n$ is a zero seq.

$\Rightarrow \{|f^n|_\alpha\}_n$ is ______.

Lem 19. A. $f_1, \ldots, f_n \in A$.

(i). $\varphi: K\langle \xi_1, \ldots, \xi_n\rangle \to A$.
$$\xi_i \mapsto f_i.$$

$\Rightarrow |f_i|_{\sup} \leq 1, \forall i$.

(Contractive under $|\cdot|_{\sup}$).

(ii). $|f_i|_{\sup} \leq 1$.

$\Rightarrow \exists!$ K-mor. (No demand of continuity).

$\varphi: K\langle \xi_1, \ldots, \xi_n\rangle \to A$ s.t.
$$\xi_i \mapsto f_i.$$

Pf: (i). ①.

(ii).

Prop 20. $\forall$ map $B \to A$ between affinoid K-algs is continuous w.r.t any res norm on A & B.

And all res norms on affinoid K-alg are equivalent.

If: $\alpha : T_m \twoheadrightarrow A$.

$\beta : T_n \twoheadrightarrow A$.

$$\begin{tikzcd} A \arrow[r, shift left, "id"] \arrow[d] & A \arrow[l, shift left, "id"] \arrow[d] \\ |\cdot|_{\alpha} & |\cdot|_{\beta} \end{tikzcd}$$

is continuous.

$\Longrightarrow$ Equivalent.

First goal:

$$B \to A. \qquad B : |\cdot|_A. \quad \alpha : T_n \to B.$$

$$\begin{tikzcd} T_n \arrow[d, "\alpha"'] \arrow[r, phantom, "\mathrm{cont}"] & {} \arrow[dr] & \\ B \arrow[rr, "\mathrm{cont}"'] & & A \end{tikzcd}$$

B. is endowed with quot top.

Example.

$$A \langle \xi \rangle.$$

$$\Big|\sum_{v \in \mathbb{N}^n} a_v \xi^v\Big| \le \max_{v \in \mathbb{N}^n} |a_v|_A.$$