

## § 3.1 Affinoid Algebras.

Def 1: A  $K$ -alg  $A$  is called an **affinoid algebra** if there exists an epic morphism of  $K$ -algebras  $\alpha: T_n \twoheadrightarrow A$  for some  $n \in \mathbb{N}$ .

Prop 2. (The category of affinoid admits **(complete) tensor product**)

Prop 3.  $A$  is an affinoid  $K$ -alg.  
(revisited. § 2.2).

(i)  $A$  is Noetherian

(ii)  $A$  is Jacobson

(iii) Noether Normalization

$\exists d \in \mathbb{Z}_{\geq 0}$   $T_d \twoheadrightarrow A$ . finite.



Def: Residue norm

$A$ : affinoid  $K$ -alg.

$\alpha: T_n \twoheadrightarrow A$ .

$|\cdot|_\alpha: A \rightarrow \mathbb{R}_{\geq 0}$ .

$f \in T_n$ .  $\frac{\alpha(f)}{\alpha(a)}$   $\mapsto$  inf  $|f - a|$   
 $a \in \ker \alpha$ .

**Proposition 5.** For an ideal  $\mathfrak{a} \subset T_n$ , view the quotient  $A = T_n/\mathfrak{a}$  as an affinoid  $K$ -algebra via the projection map  $\alpha: T_n \rightarrow T_n/\mathfrak{a}$ . The map  $|\cdot|_\alpha: T_n/\mathfrak{a} \rightarrow \mathbb{R}_{\geq 0}$  satisfies the following conditions:

- (i)  $|\cdot|_\alpha$  is a  $K$ -algebra norm, i.e. a ring norm and a  $K$ -vector space norm, and it induces the quotient topology of  $T_n$  on  $T_n/\mathfrak{a}$ . Furthermore,  $\alpha: T_n \rightarrow T_n/\mathfrak{a}$  is continuous and open.
- (ii)  $T_n/\mathfrak{a}$  is complete under  $|\cdot|_\alpha$ .
- (iii) For any  $\bar{f} \in T_n/\mathfrak{a}$ , there is an inverse image  $f \in T_n$  such that  $|\bar{f}|_\alpha = |f|$ . In particular, for any  $\bar{f} \in T_n/\mathfrak{a}$ , there is an element  $c \in K$  with  $|\bar{f}|_\alpha = |c|$ .

Pf: (i)

$$|\bar{f}|_\alpha = 0 \Leftrightarrow \bar{f} = 0 \Leftrightarrow f \in \ker \alpha.$$

( $\mathfrak{a}$  is closed).

$$\underline{|\bar{f}\bar{g}| \leq |\bar{f}| \cdot |\bar{g}|}$$

$$\begin{aligned} |f|_2 &= \inf_{a \in \mathbb{R}} |f - a| \\ &= |f - a| \quad \text{for some } a \in \mathbb{R}. \end{aligned}$$

$$|g|_2 = |g - b|, \quad b \in \mathbb{R}$$

$$\begin{aligned} |fg|_2 &\leq |(f-a)(g-b)| \\ &= |f-a| \cdot |g-b| \\ &= |f|_2 \cdot |g|_2 \end{aligned}$$

---

$$|f+g|_2 \leq \max\{|f|_2, |g|_2\}.$$

$$|cf|_2 = |c| \cdot |f|_2. \quad \text{clear.}$$



Openness of  $\alpha$ :

$$\alpha(B(0, \varepsilon)) = B_2(0, \varepsilon).$$

$\hookleftarrow$ : Simple

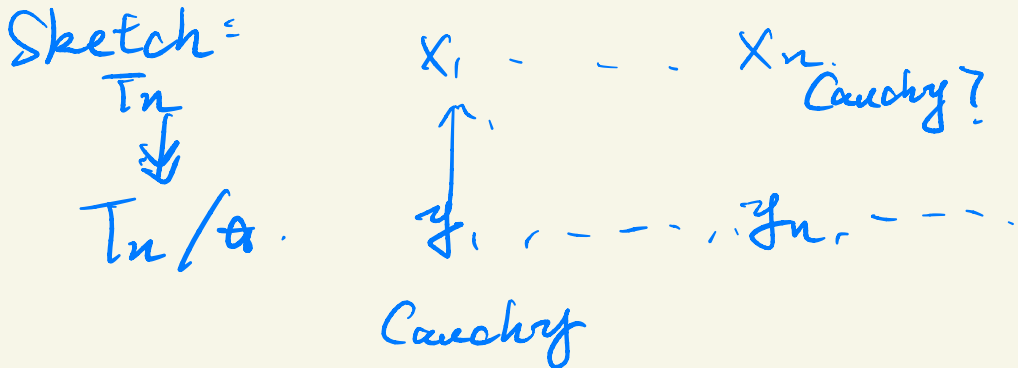
$\supset$ :  $\alpha$  is strictly closed.

Continuity:

$$\underline{|\alpha(A)|_2 \leq |A|}.$$

(ii)  $T_n(\mathfrak{a})$  is complete under  $|\cdot|_2$ .

Sketch:



$$\underline{|x_{n+1} - x_n| = |y_{n+1} - y_n|_2.}$$

(iff),  $|f|_a = |f|_{T_n} = |c|$  for some  $c \in K$ .

$\checkmark$  det.  $c \in K$ .

$|K|$ .


$\alpha: T_n \rightarrow A$ . ker  $\alpha = \mathfrak{a} \subseteq T_n$

①  $| \cdot |_a$ .

②  $f: V(\mathfrak{a}) \subset \underline{B^n(K)}$ .

$\begin{matrix} \uparrow \\ A \end{matrix} \rightarrow \overline{K}$

$$|f|_{\text{sup}} = \sup_{x \in \text{Max } A} |f(x)|$$

$f(x) \in A/\mathfrak{a} \iff$  

$\uparrow$   
 field over K.

$|f|_{\text{sup}} = 0 \iff f \in A$  is nilpotent.

Prop 6: Power multiplicative:

$$|f^n|_{\text{sup}} = |f|_{\text{sup}}^n, \quad \forall f \in A, \\ \forall n \in \mathbb{Z}_{\geq 1}.$$

Prop 7:  $\varphi: B \rightarrow A$  is a morphism between affine  $K$ -algebras.

Then  $|\varphi(b)|_{\text{sup}} \leq |b|_{\text{sup}}, \quad \forall b \in B$ .

Pf: 
$$|\varphi(b)|_{\text{sup}} \leq |b|_{\text{sup}}.$$
  
 $m \in \text{Max } A \quad \exists n \in \text{Max } B.$

Choose  $n = \varphi^{-1}(m) \in B$ .

$n$  is maximal.

$$K \hookrightarrow B/n \hookrightarrow A/m.$$

is a field

finite ext  
over  $K$ .

$$|\varphi(b)(m)| = |b(m)|. \quad \forall m \in \max A$$

$$\Rightarrow |\varphi(b)|_{\text{sup}} \leq |b|_{\text{sup}}. \quad \square$$

Prop 8.  $|\cdot|_{\text{sup}} = |\cdot|$  (Gauss norm)

on a Tate algebra  $T_n$ .

If:  $f \in T_n$ .

$$|f| = \max \{ |f(x)| : x \in B^n(\overline{K}) \}$$

(maximum principle)

$$= |f|_{\text{sup}}.$$

Recall:

$$B^n(\overline{K})$$

$$\rightarrow \max T_n.$$

$$x = (x_1, \dots, x_n) \mapsto \max.$$

$$= \text{ker}(T_n \rightarrow \overline{K})$$

$$f \mapsto f(x_1, \dots, x_n)$$

Prop 1.  $A$ : affine  $K$ -alg.

$$\alpha: T_n \twoheadrightarrow A.$$

$(\cdot)_\alpha$ : Residue norm.

Then  $\|f\|_{\text{sup}} \leq \|f\|_\alpha, \forall f \in A.$

$\Rightarrow \|f\|_{\text{sup}}$  is finite.

Pf: Choose  $m \in \max A.$  and  $f \in A.$   
|  
 $n = \alpha^{-1}(m) \in \max T_n.$

$f \in A$  pre- $\alpha$   $g \in T_n.$   $\|g\| = \|f\|_\alpha.$

$$\underline{\underline{|f(m)| = |g(n)| \leq \|g\| = \|f\|_\alpha.}}$$

$\square$

Prop 10.  $\|f\|_{\text{sup}} = 0$



$f$  is nilpotent.

iff:  $A$  is Jacobson.  $\square$

- $\|\cdot\|_{\text{sup}}$  &  $\|\cdot\|_2$ .

Lem 11.

$$p(\mathcal{Y}) = \mathcal{Y}^r + c_1 \mathcal{Y}^{r-1} + \dots + c_r \in K[\mathcal{Y}]$$
$$= \prod_{j=1}^r (\mathcal{Y} - \alpha_j), \quad \alpha_j \in \overline{K}.$$

Then  $\max_{j=1, \dots, r} |\alpha_j| = \max_{i=1, \dots, r} |c_i|^{\frac{1}{i}}$ .

iff:  $\geq$ :

$$c_i = \sum_{m_1 < \dots < m_i} \alpha_{m_1} \dots \alpha_{m_i}$$

$$|c_i| \leq \max \{ |\alpha_{m_1}| \dots |\alpha_{m_i}| \}.$$

$$\leq \left( \max_{\delta=1, \dots, r} |\alpha_\delta| \right)^i$$

$\leq$  :

$$I \subset \{1, \dots, r\} \quad \text{s.t.}$$

$$|\alpha_i| = \max_{\delta=1, \dots, r} |\alpha_\delta| \iff i \in I.$$

$$t = |I|$$

$$|C_t| = \left| \sum_{\varepsilon_1, \dots, \varepsilon_t} \alpha_{\varepsilon_1} \cdots \alpha_{\varepsilon_t} \right|.$$

$$= \left| \prod_{\varepsilon \in I} \alpha_\varepsilon \right| = \left( \max_{\delta=1, \dots, r} |\alpha_\delta| \right)^t.$$

$$\Rightarrow |C_t|^{\frac{1}{t}} = \max |\alpha_\delta|. \quad \square$$

$\sigma(p)$ .  $p \in A[\mathcal{S}]$ .  $A = \text{normed ring}$ .

$$\sigma(p) = \max_{i=1, \dots, r} |c_i|^{1/i} \quad (\text{Spectral value})$$

where  $p = \mathcal{S}^r + c_1 \mathcal{S}^{r-1} + \dots + c_r$ .

$$\|p\| = \max |c_i| = \sigma(p).$$

Lemma 12:  $A$  is a (semi-)normed ring.

$p, q \in A[\mathcal{S}]$  are monic.

$$\sigma(pq) \leq \max(\sigma(p), \sigma(q)).$$

Proof:  $p = \sum_{i=0}^m a_i \mathcal{S}^{m-i}$       $a_m = 1$ .

$$q = \sum_{j=0}^n b_j \mathcal{S}^{n-j}$$

$$pq = \sum_{\lambda=0}^{m+n} c_\lambda \mathcal{S}^{m+n-\lambda} \quad c_\lambda = \sum_{i+j=\lambda} a_i b_j.$$



$$|C_{\lambda}| \leq \max(\sigma(A), \sigma(B))^{\lambda}.$$

We know

$$|a_i| \leq \sigma(A)^i \leq \max(\sigma(A), \sigma(B))^i$$

$$|b_j| \leq \sigma(B)^j \leq \max(\sigma(A), \sigma(B))^j.$$

$$|C_{\lambda}| = \left| \sum_{i+j=\lambda} a_i b_j \right|$$

$$\leq \max_{i+j=\lambda} |a_i| |b_j|$$

$$\leq \max_{i+j=\lambda} \left( \max(\sigma(A), \sigma(B)) \right)^{i+j}.$$

□.

Lemma 3.  $T_d \hookrightarrow A$ : finite monic of  $K$ -alg.

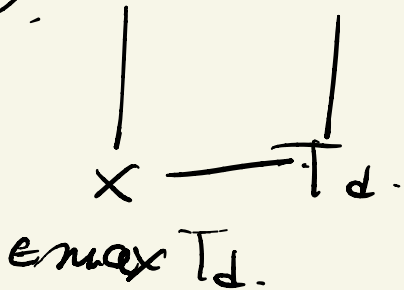
$A$  is torsion-free as  $T_d$ -mod  
 $f \in A$ .

(i)  $\exists!$  monic  $P_f = y^n + a_1 y^{n-1} + \dots + a_n$   
 $\in T_d[y]$  of minimal deg s.t.  $P_f(f) = 0$ .

And  $\ker [T_d[y] \rightarrow A]$   
 $[y \mapsto f]$  is

generated by  $P_f$  as a  $T_d$ -hom.

(ii)  $y_1, \dots, y_s \in \max A$



Then

$$\max_{j=1, \dots, s} |f(y_j)|$$

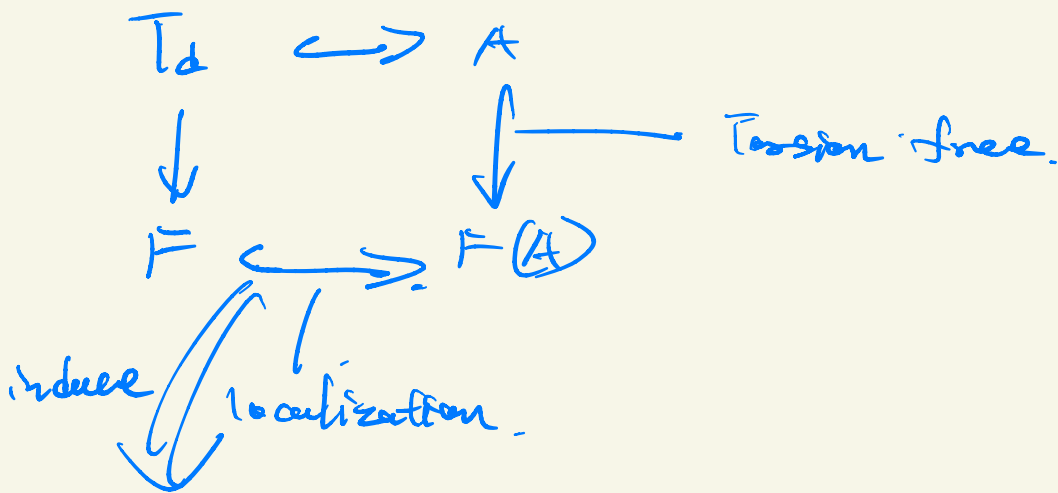
$$= \max_{i=1, \dots, r} |a_i(x)|^{1/n}$$

$\langle \text{civ} \rangle \left( \leftarrow \text{civ} \right)$  by taking sup over  $\text{Max } \mathbb{I}$

$$\|f\|_{\text{sup}} = \max_{i=1, \dots, r} |a_i| \stackrel{\frac{1}{i}}{\text{sup}}$$

Pf.  $F = \mathbb{Q}(\mathbb{I}_d)$ .  $F(A) = A \otimes_{\mathbb{I}_d} F$

$\exists$  diag of inclusions.



$$\text{ker} \left( \begin{array}{ccc}
 F[\mathbb{I}] & \twoheadrightarrow & F(A) \\
 \mathbb{I} & \mapsto & \mathbb{I}
 \end{array} \right) = (P_f) \quad P_f \in F[\mathbb{I}]$$

Claim:  $P_f \in T_d[\mathcal{S}]$ .

$T_d \hookrightarrow A$  finite.

$f \in A \Rightarrow h \in T_d[\mathcal{S}]$  ( $\in F[\mathcal{S}]$ ).  
 $h(f) = 0$ .  $\swarrow$   $h$  is monic.

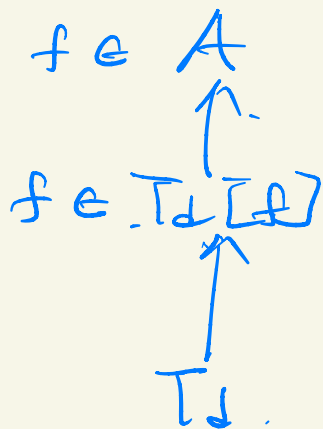
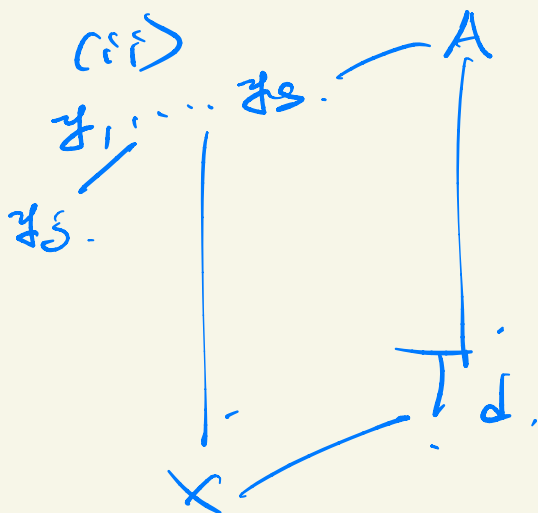
$\Rightarrow P_f / h$  over  $F[\mathcal{S}]$ .

$\Rightarrow P_f / h$  over  $T_d[\mathcal{S}]$ , as  $T_d$  is factorial.

( $h \in T_d[\mathcal{S}]$ ,  $h(f) = 0 \Rightarrow P_f / h$ ).

Thus  $T_d[\mathcal{S}] \rightarrow A$  has kernel  
 $\mathfrak{S} \mapsto \mathfrak{d}$ .

( $P_f$ ).



$$\text{Max } A \rightsquigarrow \text{Max } T_d[f] \rightsquigarrow \text{Max } T_d$$

$$y_\delta \text{ --- } z_i \text{ --- } x$$

$$|f(y_\delta)| = |f(z_i)|$$

$$T_d/x \hookrightarrow T_d[f]/z_i \hookrightarrow A/y_\delta$$

$$\underline{f} \qquad \underline{f}$$

$$\max |f(y_\delta)| = \max |f(z_i)|$$

Thus

$$\text{WLOG, } A = T_d[f]$$

$$A = T_d[f] \stackrel{\text{by (i)}}{=} T_d[\mathcal{S}] / (P_f).$$

$$\begin{array}{ccc} y_1, \dots, y_r & \xrightarrow{T_d[f]} & T_d[\mathcal{S}] / (P_f) \\ & & \downarrow A \\ & & T_d \\ \downarrow & \longrightarrow & \downarrow \\ X & & T_d \end{array}$$

Set  $L = T_d/X$  (is finite over  $K$ ).

$$A/X = \mathcal{L}[\mathcal{S}] / (\overline{P_f}).$$

$\overline{P_f}$  has roots in  $\overline{L}$ , denoted by  $\alpha_1, \dots, \alpha_r$

$$\begin{aligned} \underline{\max_{\delta} |f(y_j)|} &= \max_{j=1, \dots, r} |\alpha_j|. \\ &\quad \downarrow \overline{P_f} \\ &= \max_i |\alpha_i(x)|^{\frac{1}{i}} \quad \square \end{aligned}$$

Lem 14. Let  $\varphi: B \rightarrow A$  is finite hom  
between affinoid  $K$ -algebras.

Then for any  $f \in A$ ,  $\exists$  equation.

$$f^n + b_1 f^{n-1} + \dots + b_r = 0, \quad b_i \in B.$$

$$\|f\|_{\text{sup}} = \max_{i=1, \dots, r} \|b_i\|_{\text{sup}}^{\frac{1}{i}}$$

Thm 15.  $A$  is an affine  $K$ -algebra.

$f \in A$ .

Then  $\exists$  reflex  $A$  s.t.

$$|f(x)| = |f|_{\text{sup}}.$$

If:  $A$  minimal primes  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ .

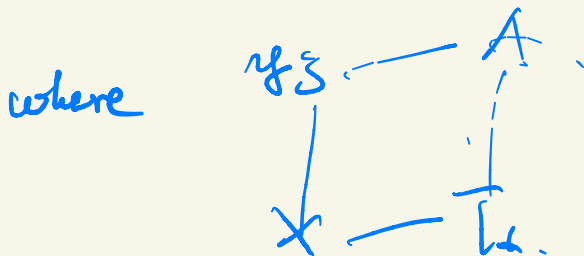
WLOG.  $A$  is an int domain.

$\Rightarrow$  A finite morph  $T_d \hookrightarrow A$ .

Choose  $f \in A$ .

$f^n + a_1 f^{n-1} + \dots + a_n = 0$  over  $T_d$ .

$$\Rightarrow \max_{j=1, \dots, s} |f(y_{j0})| = \max_{i=1, \dots, s} |a_i(x)|^{1/i}$$





$$\underbrace{|a_1 \dots a_r(x)|}_{\prod |a_i(x)|} = \underbrace{|a_1 \dots a_r|}_{|a_1| \dots |a_r|}$$

for some  $x \in \max T_d$ .

$$\Rightarrow |a_i(x)| = |a_i|, \forall i.$$

$$\begin{aligned} \max_{y \in T_d} |f(y)| &= \max_{i=1, \dots, r} |a_i(x)|^{\frac{1}{i}} \\ &= \max_{i=1, \dots, r} |a_i|^{\frac{1}{i}} \\ &= \|f\|_{\text{sup}} \quad (\text{Lem 4}). \end{aligned}$$

□

Prop 16.,  $A$ : Affine  $K$ -alg

fca.

$\Rightarrow \exists n \in \mathbb{Z}_{>0}$  s.t.  $\|f\|_{\text{sup}}^n \in |K|$

Pf: Lem 14:  $T_d \xrightarrow{\text{finite}} A$ .

$$\|f\|_{\text{sup}} = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq r}} \|b_i\|_{\text{sup}}$$

$$= \|b\|_{\text{sup}}^{\frac{r}{s}}$$

$$\|f\|_{\text{sup}}^{\frac{s}{r}} = \|b\|_{\text{sup}} \in |K|.$$

Thm 17.  $f \in A$ ,  $\|f\|_A = 1$ .

(i)  $\|f\|_{\infty} \leq 1$ .

(ii)  $\exists f^r + a_1 f^{r-1} + \dots + a_r = 0$  with  
coefficients  $a_i \in A$  s.t.  $\|a_i\|_A \leq 1$ .

(iii)  $\|f^r\|_A$  is bounded.

Sketch:  $T_d \xrightarrow{\text{finite}} A$

(i)  $\Rightarrow$  (ii) by Lem 14:

$\exists$  equation with  $\|a_i\|_{\infty} = \|a_i\|_A$ .

where  $a_i \in T_d$ .

$T_d \xrightarrow{\alpha} \frac{T_n}{\alpha} \xrightarrow{\beta} A$ .

Contradictive  $\Rightarrow \|a_i\|_A \leq 1$ .

$$(f_i) \Rightarrow (f_i i).$$

$$A^\circ = \{g \in A : |g|_a \leq 1\}.$$

$(f_i) \Rightarrow$   $A[A]$  is finite over  $A^\circ$ .  
norms are bounded

$\Rightarrow |f^n|_a$  is bounded.

$$(f_i) \Rightarrow (c_i). \quad |f|_{\text{sup}}^n = \underbrace{|f|_{\text{sup}}^n}_{\leq |f^n|_a} \quad \forall n \geq 1.$$

$$\Rightarrow |f|_{\text{sup}} \leq 1.$$

Cor 16. TFAE.  $\alpha = T_n \Rightarrow A$ .

(i)  $|f|_{\text{sup}} < 1$ .

(ii)  $\{f^n\}_n$  is a zero seq.

Sketch: (i)  $\Rightarrow$  (ii).

$$|f^n|_{\text{sup}} = |\alpha|^n. \quad c \in K.$$

$$\Rightarrow |c^{-1} f^n|_{\text{sup}} = 1.$$

Thm 17.  $\Rightarrow \{ |c^{-n} f^{rn}|_n \}$  is bounded.

$\Rightarrow \{ |f^{rn}|_n \}$  is a zero seq.

$\Rightarrow \{ |f^n|_n \}$  is \_\_\_\_\_.

Lem 19.  $A, f_1, \dots, f_n \in A.$

$$(i) \varphi = K \langle \xi_1, \dots, \xi_n \rangle \rightarrow A.$$
$$\xi_i \mapsto f_i.$$

$$\Rightarrow \|f_i\|_{\text{sup}} \leq 1, \forall i.$$

(Contractive under  $\|\cdot\|_{\text{sup}}$ ).

$$(ii) \|f_i\|_{\text{sup}} \leq 1.$$

$\Rightarrow \exists!$   $K$ -norm. (No demand of continuity).

$$\varphi = K \langle \xi_1, \dots, \xi_n \rangle \rightarrow A \quad s-t.$$
$$\xi_i \mapsto f_i.$$

$\exists!$  (i).  $\square$ .

(ii).

Prop 20.  $\forall$  map  $B \rightarrow A$  between  
affine  $K$ -algs is continuous w.r.t  
any res norm on  $A$  &  $B$ .

And all res norms on affine  
 $K$ -algs are equivalent.

$$\text{If: } \begin{array}{l} \alpha: T_m \rightarrow A \\ \beta: T_n \rightarrow A \end{array}$$

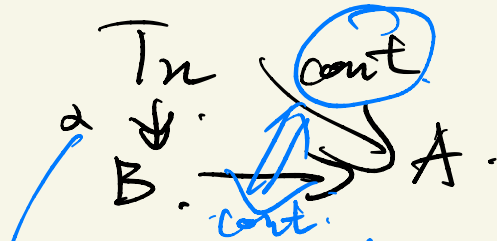
$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ \downarrow & \xleftarrow{\text{id}} & \downarrow \\ 1 \cdot 2 & & \downarrow \beta \end{array} \quad \text{is continuous.}$$

$\Rightarrow$  Equivalent.

Final para:

$$B \rightarrow A.$$

$$B: 1 \times 2. \quad \alpha = T_n \rightarrow B.$$



B. is endowed with exact top.

□.

Example.

$$A \langle \mathfrak{S} \rangle.$$

$$\left| \sum a_{\nu} \xi^{\nu} \right|_{\mathfrak{W}^n} \Leftarrow \max_{\nu \in \mathfrak{W}^n} |a_{\nu}|_2.$$



