Affinoid subdomains.

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§1 Affinoid spaces (recap)

K non-arch local field.

Tate alg $T_n = K\langle \xi_1, \cdots, \xi_n \rangle \supseteq \mathfrak{a}$ ideal.

$\rightsquigarrow T_n/\mathfrak{a} = A$ affinoid algebra.

Def'n (1) The <u>affinoid K-space</u> is the datum

$$\mathrm{Sp}\,A := (\mathrm{Max}\,A, A)$$

<u>note</u> Algebraic geom: $\mathrm{Spec}\,A \leftrightarrow$ prime ideals

Rigid geom: $\mathrm{Sp}\,A\ (=\mathrm{Spm}\,A) \longleftrightarrow$ max'l ideals

(2) The <u>vanishing locus</u> of $\mathfrak{a}$ is

$$V(\mathfrak{a}) := \{x \in \mathrm{Sp}\,A : f(x) = 0,\ \forall f \in \mathfrak{a}\}$$
$$= \{x \in \mathrm{Sp}\,A : \mathfrak{a} \subseteq x\}.$$

(3) The <u>basic open subset</u> is

$$D_f := \{x \in \mathrm{Sp}\,A : f(x) \neq 0\},\quad f \in A.$$

Rmk How about $V(\mathfrak{a}) = \{x \in \mathbb{B}^n(\bar{K}) : \forall f \in \mathfrak{a},\ f(x) = 0\}$?

- $f \in A \rightsquigarrow$ function $\mathrm{Sp}\,A \to \bar{K}$.
- $f \in T_n \rightsquigarrow$ function $\mathbb{B}^n(\bar{K}) \to \bar{K}$.

T_n is $\mathbb{B}^n(\bar{K})/\mathrm{Gal}(\bar{K}/K) \to \mathrm{Max}\,T_n \twoheadrightarrow \mathrm{Max}\,A$

Prop (1) $\mathfrak{a} \subseteq \mathfrak{b} \Rightarrow V(\mathfrak{a}) \supseteq V(\mathfrak{b})$; $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$; $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(2) $\{D_f\}_{f \in A}$: Zariski top basis on $\mathrm{Sp}\,A$.

(3) (Hilbert Nullstellensatz).

$Y \subseteq \mathrm{Sp}\,A$, define $\mathrm{id}(Y) := \bigcap_{y \in Y} \mathfrak{m}_y \subseteq A = \{f \in A : f(y) = 0,\ \forall y \in Y\}$.

Then $V(\mathrm{id}(Y)) = \bar{Y}$, $\mathrm{id}(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Cor $f_1, \dots, f_n \in A$. Then $(f_1, \dots, f_n) = A \iff \bigcup D_{f_i} = \mathrm{Sp}\, A$.

Thm There's an equivalence of cats:

$$\{\text{affinoid } K\text{-algebras}\}^{\mathrm{opp}} \simeq \{\text{affinoid } K\text{-spaces}\}$$
$$A \longleftrightarrow \mathrm{Sp}\, A.$$

E.g. (1) $\sigma : A \longrightarrow B \rightsquigarrow \mathrm{Sp}\, B \longrightarrow \mathrm{Sp}\, A$.

$$\begin{array}{ccccc} \cup\!\! \vert & & \cup\!\! \vert & & \\ \boxed{\sigma^{-1}(m)} & & m & & m \longmapsto \sigma^{-1}(m). \end{array}$$

still max'l $\to$ $\boxed{\sigma^{-1}(m)}$

Also, $K \hookrightarrow A/\sigma^{-1}(m) \hookrightarrow B/m$ monomorphisms.

(2) Functoriality: e.g.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ T \arrow[r] & S \hat{\otimes}_R T \end{tikzcd} \longleftrightarrow \begin{tikzcd} \mathrm{Sp}\, R & \mathrm{Sp}\, S \arrow[l] \\ \mathrm{Sp}\, T \arrow[u] & \mathrm{Sp}(S \hat{\otimes}_R T) \arrow[l] \arrow[u] \end{tikzcd}$$

$$\mathrm{Sp}(S \hat{\otimes}_R T) = \mathrm{Sp}\, S \times_{\mathrm{Sp}\, R} \mathrm{Sp}\, T.$$

§2 Canonical topology

Motivation Rigid space: $X = \mathrm{Sp}\, A$ + Grothendieck top $\begin{cases} \text{adm opens} \leftarrow \text{affinoid subdomains} \\ \text{adm open coverings} \end{cases}$

$\rightsquigarrow$ sheaf $\mathcal{O}_X$ of rigid analytic functions s.t. $A = $ "$\Gamma(X, \mathcal{O}_X)$".

$\rightsquigarrow$ non-arch analogue $/ \overline{\mathbb{C}_p} \overset{?}{\simeq} \mathbb{C}$ of $\underbrace{\text{alg geom}}_{\text{Zar top}} / \mathbb{C}$ or $\underbrace{\text{arith geom}}_{\text{conf top / Zar top. \& disc.}} / \mathbb{Q}$

canonical top / Zar top & non-arch norms

e.g. rigid-analytic GAGA, coh comparison, uniformizations of Shimura varieties, etc.

Example $X = \mathrm{Sp}\, T_1 = \mathrm{Sp}\, K\langle \zeta \rangle$ closed unit disc.

$Y = U \sqcup S$, U = open unit disc, $S = \mathrm{Sp}\, k\langle \zeta, \zeta^{-1}\rangle$ boundary.

$\rightsquigarrow Y \to X$ open bijective immersion, but $Y \not\cong X$ as aff spaces.

b/c $U \neq \bigcup^{\infty}$ (aff subdomains)

$\Rightarrow \{U, S\} \neq$ adm open cover under Grothendieck top.

as a set $\quad (\mathrm{Max}\, T_n, T_n)$

Fix $T_n \twoheadrightarrow A \rightsquigarrow \mathrm{Sp}\, A \hookrightarrow \mathrm{Sp}\, T_n = \underbrace{\mathrm{Max}\, T_n}_{\text{as a top space}} \twoheadleftarrow \mathbb{B}^n(\bar{k})$

$\mathbb{A}^n_{\bar{k}}$-top $\rightsquigarrow \mathbb{B}^n(\bar{k})$-top $\rightsquigarrow \mathrm{Sp}\, T_n$-top $\rightsquigarrow \mathrm{Sp}\, A$-top

- indep't of the Zariski top & the choice of $\alpha : T_n \twoheadrightarrow A$.
- inherited from $\bar{k}$-top (intrinsic).

For $\varepsilon > 0$, $f \in A$, $X = \mathrm{Sp}\, A$, define

$$X(f;\varepsilon) := \{x \in X : |f(x)| \leq \varepsilon\},$$

$$X(f) := X(f, 1),$$

$$X(f_1, \cdots, f_r) := X(f_1) \cap \cdots \cap X(f_r).$$

<u>Def'n</u> A <u>canonical top</u> on $\mathrm{Sp}\, A$ is the top gen'd by $\{X(f;\varepsilon)\}_{f \in A, \varepsilon > 0}$ as open subsets.

<u>Prop</u> $X = \mathrm{Sp}\, A$, can top on X is gen'd by $\{X(f)\}_{f \in A}$.

i.e. $\forall U \subseteq X$ open, $U = \bigcup U_i$, $U_i = X(f_{i_1}, \cdots, f_{i_{n(i)}})$.

<u>Proof</u> $X(f;\varepsilon) = \bigcup_{\substack{\varepsilon' \leq \varepsilon \\ \varepsilon' \in |\bar{k}^\times|}} X(f;\varepsilon')$ $\quad x \in \mathrm{Max}\, A$, $(A/x)/k$, $\quad A \xrightarrow{\pi} A/x$, $f \longmapsto f(x)$

$(|f(x)| \in |\bar{k}^\times|, \forall x \in k)$.

$\forall \varepsilon' \in |\bar{k}^\times|$, $\exists s$ s.t. $(\varepsilon')^s \in |k^\times|$ by def'n.

$\Rightarrow X(f;\varepsilon') = X(f^s;(\varepsilon')^s) = X(c^{-1}f^s;1) = X(c^{-1}f^s)$. $\square$

$|f(x)^s| = |f(x)|^s \leq (\varepsilon')^s = |c| \Leftrightarrow |c^{-1}f(x)^s| = \frac{1}{c}|f(x)|^s \leq 1$.

Lem Let $X = \mathrm{Sp}A \ni x$, $f \in A$, $|f(x)| = \varepsilon > 0$.

Then $\exists g \in A$ s.t. $g(x) = 0$, and $(y \in X(g) \Rightarrow |f(y)| = \varepsilon)$.

In particular, $X(g) \subseteq \{y \in X : |f(y)| = \varepsilon\}$ is an open nbhd of x.

Proof Consider $A \longrightarrow \boxed{A/m_x} \hookleftarrow / K$

$f \longmapsto \bar{f} = f(x)$.

$\rightsquigarrow P(\xi) = \xi^n + c_1\xi^{n-1} + \cdots + c_n \in K[\xi]$ min poly of $f(x) \in A/m_x$ over K.

$= \prod_{i=1}^{n} (\xi - \alpha_i)$, $\alpha_i \in \bar{K}$

$\Rightarrow \forall 1 \le i \le n$, $\varepsilon = |f(x)| = |\alpha_i|$ by def'n of $|\cdot|$ on A/m_x.

Take $g = P(f) \in A$.

Check: (1) $g(x) = P(f(x)) = P(\bar{f}) = 0$.

(2) $\forall y \in X$, $|g(y)| \le 1 \Rightarrow_{?} |f(y)| = \varepsilon$

Claim $|g(y)| < \varepsilon^n \Rightarrow |f(y)| = \varepsilon$.

Otherwise. $|f(y) - \alpha_i| = \max(|f(y)|, |\alpha_i|) \ge \varepsilon$.

$|f(y)| \ne |\alpha_i|$

But $|g(y)| = |P(f(y))| = \prod |f(y) - \alpha_i| \ge \varepsilon^n$, contradiction.

So $\exists c \in K^\times$, $|c| < \varepsilon^n$. s.t. $X(c^{-1}g) \subseteq \{y \in X : |f(y)| = \varepsilon\}$.

(may replace g with $c^{-1}g$ if necessary). □

Cor (1) The following are open w.r.t. can top:

$\{x \in \mathrm{Sp}A : |f(x)| = \varepsilon \text{ (resp. } \le \varepsilon, \ge \varepsilon)\}$.

$\{x \in \mathrm{Sp}A : f(x) \ne 0\} = \{x \in \mathrm{Sp}A : |f(x)| \ne 0\}$.

(2) $X = \mathrm{Sp}A$, $\forall x \in X$, $X(f_1, \ldots, f_r)$'s with $f_1, \ldots, f_r \in m_x$

form a basis of nbhd of x.

Prop $\varphi : \mathrm{Sp}B \to \mathrm{Sp}A$, $\varphi^* : A \to B$ morphisms, $f_1, \dots, f_r \in A$.

Then $\varphi^{-1}(\underbrace{(\mathrm{Sp}A)(f_1, \dots, f_r)}_{\text{open of Sp}A}) = \underbrace{\mathrm{Sp}(B)(\varphi^*(f_1), \dots, \varphi^*(f_r))}_{\text{open of Sp}B}$.

In particular, any morphism φ is continuous.

Proof $\forall y \in \mathrm{Sp}B$,

$$\begin{tikzcd}
\varphi(y) \in \mathrm{Sp}A \arrow[d, leftrightarrow] \arrow[dd, bend right=60] & \mathrm{Sp}B \ni y \arrow[l, "\varphi"'] \arrow[d, leftrightarrow] \arrow[dd, bend left=60] \\
f \in A \arrow[r, "\varphi^*"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \ni \varphi^* f \arrow[d] \\
A/\mathfrak{m}_{\varphi(y)} \arrow[r, hook] & B/\mathfrak{m}_y \\
& K
\end{tikzcd}$$

$\Rightarrow |f(\varphi(y))| = |(\varphi^* f)(y)|$, $\forall f \in A$

$\Rightarrow \varphi^{-1}((\mathrm{Sp}A)(f)) = (\mathrm{Sp}B)(\varphi^*(f))$. $\square$

§3 Affinoid subdomains

Def'n (Spatial aff subdomains) $X = \mathrm{Sp}A$.

(1) Weierstrass dom: $X(f_1, \dots, f_r) = \{x \in X : |f_i(x)| \le 1\}$

(2) Laurent dom: $X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{x \in X : |f_i(x)| \le 1, |g_j(x)| \ge 1\}$.

(3) Rational dom: $X\left(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0}\right) = \{x \in X : |f_i(x)| \le |f_0(x)|\}$

where $f_0, \dots, f_r$ has no common zeros, i.e. $(f_0, \dots, f_r) = A$.

Rmk In (3), $(f_0, \dots, f_r) = A$ cannot be dropped.

E.g. $X = \mathrm{Sp}\, T_1 = \mathrm{Sp}\, K\langle \zeta \rangle$,

$\rightsquigarrow X\left(\frac{\zeta}{c\zeta}\right) = \{x \in X : |\zeta(x)| \le \underbrace{|c\zeta(x)|}_{= |c| \cdot |\zeta(x)|} \text{ for } c \in K, 0 < |c| < 1\}$

$= \{x \in X : |\zeta(x)| = 0\}$

$= \mathrm{pt}$, cannot be open.

Lem Special doms are open w.r.t. can top.

Proof Weierstrass/Laurent: done.

Rational: $\forall x \in X(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0})$, $|f_0(x)| \neq 0$,

$\exists$ open nbhd $U = X(f_1; f_1(x)) \cap \dots \cap X(f_r; f_r(x)) \cap \{y : |f_0(y)| \geq |f_0(x)|\}$.

(finite intersection of opens)

Check: $x \in U$ obvious. $\forall y \in U$, $|f_i(y)| \leq |f_i(x)|$, $|f_0(y)| \geq |f_0(x)|$

$\Rightarrow |f_i(y)| \leq |f_0(y)| \Rightarrow y \in X(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0})$. $\square$

Def'n $X = \mathrm{Sp} A$. A subset $U \subseteq X$ is an <u>affinoid subdomain</u> if

(i) $\exists\, \iota : X' \to X$ s.t. $\mathrm{im}\, \iota \subseteq U$,

(ii) $\forall\, \varphi : Y \to X$ s.t. $\varphi(Y) \subseteq U$, $\exists!\, Y \xrightarrow{f} X'$ s.t.

$$\begin{tikzcd} Y \arrow[dr, dashed, "\exists!"'] \arrow[drr, bend left, "\varphi"] & & \\ & X' \arrow[r, "\iota"'] & X \end{tikzcd}$$

Lem $X = \mathrm{Sp} A$, $X' = \mathrm{Sp} A'$, $U \subseteq X$ aff subdom $\Big\}$ as above

$(\iota, \iota^*) : X' \to X$, $\iota^* : A \to A'$.

Then (1) ι injective, $\iota(X') = U$ ($\rightsquigarrow$ bijection $X' \xrightarrow{\sim} U$).

(2) $\forall x \in X'$, $n \in \mathbb{N}$, $A/m^n_{\iota(x)} \xrightarrow{\sim} A'/m^n_x$.

(3) $\forall x \in X'$, $m_x = m_{\iota(x)} A'$.

Proof $\forall y \in U$,

$$\begin{tikzcd} A \arrow[r, "\iota^*"] \arrow[d, "\pi"'] & A' \arrow[d, "\pi'"] \arrow[dl, dashed, "\alpha"'] \\ A/m_y^n \arrow[r, "\sigma"] & A'/m_y^n A' \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \mathrm{Sp} A & \mathrm{Sp} A' \arrow[l, "\iota"'] \\ \mathrm{Sp}(A/m_y^n) \arrow[u] \arrow[ur, dashed] & \mathrm{Sp}(A'/m_y^n A') \arrow[l] \arrow[u] \end{tikzcd}$$

Notice that $\mathrm{Max}\, A/m_y^n = \{m_y\} = \mathrm{Sp}\, A/m_y^n$.

$\Rightarrow \mathrm{im}\big(\mathrm{Sp}(A/m_y^n) \to \mathrm{Sp} A\big) = \{y\} \subseteq U$.

$\Rightarrow \exists!\ \mathrm{Sp}\,A/\mathfrak{m}_y^n \longrightarrow \mathrm{Sp}\,A'$ by univ property

$\Rightarrow \exists!\ \alpha : A' \longrightarrow A/\mathfrak{m}_y^n$ s.t. $\pi = \alpha \circ \iota^*$, $\pi' = \sigma \circ \alpha$.

Now π surj $\Rightarrow \alpha$ surj, π' surj $\Rightarrow \sigma$ surj.

Have $\ker \pi' = \mathfrak{m}_y^n A' \subset \ker \alpha \Rightarrow \ker \sigma = 0 \Rightarrow \sigma$ inj.

For $n = 1$, $\mathfrak{m}_{\iota(x)} A' = \mathfrak{m}_x \Rightarrow (1)(3)$.

And σ bijective $\Rightarrow (2)$. $\square$

<u>Prop</u> $X = \mathrm{Sp}\,A$. Special aff subdoms are truely aff subdoms.

<u>Lem</u> (For checking $\mathrm{im}\,\iota \subseteq U$).

Given $\varphi : Y \to X$, $\forall y \in Y$, $f_1, \dots, f_r \in A$,

we have $\varphi(Y) \subseteq X(f_1, \dots, f_r) \iff \forall i,\ |f_i(\varphi(y))| \le 1$

$$|f_i(\varphi(y))| = |(\varphi^* f_i)(y)|$$

(by max principle) $\iff |\varphi^* f_i|_{\sup} \le 1$, $\forall i$.

<u>Proof</u> (a) Weierstrass: $f := (f_1, \dots, f_r) \rightsquigarrow X(f) \subseteq X$.

Let $A\langle f \rangle = A\langle f_1, \dots, f_r \rangle = A\langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_1 - f_1, \dots, \zeta_r - f_r)$.

$\rightsquigarrow \iota^* : A \longrightarrow A\langle f \rangle$, $\iota : \mathrm{Sp}\,A\langle f \rangle \longrightarrow X$.

<u>Check</u> (1) $\mathrm{im}\,\iota \subseteq \mathrm{Sp}\,X(f)$: for some fixed $\alpha : T_n \to A$,

$$|\iota^*(f_i)|_{\sup} \le |\zeta_i|_{\sup} \le |\zeta_i|_\alpha \le 1.$$

(2) Univ property:

Assume $\varphi : Y = \mathrm{Sp}\,B \to X$ s.t. $\varphi(Y) \subseteq X(f) \rightsquigarrow \varphi^* : A \to B$

$\rightsquigarrow$ extension $\tilde{\varphi}^* : A\langle f \rangle \longrightarrow B \quad \Rightarrow \zeta_i - f_i \in \ker \tilde{\varphi}^*$.

$$\zeta_i \longmapsto \varphi^*(f_i)$$

$\Rightarrow$

$$\begin{tikzcd} & A\langle f \rangle \arrow[d, dashed, "\tilde{\varphi}^*"] \\ A \arrow[ur, "\iota^*"] \arrow[r, "\varphi^*"'] \arrow[ur, phantom, shift right=4, "\circlearrowleft"] & B \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Sp}\,A\langle f \rangle \arrow[dl, "\iota"'] \\ \mathrm{Sp}\,A \arrow[ur, phantom, shift right=4, "\circlearrowleft"] & \mathrm{Sp}\,B \arrow[l, "\varphi"] \arrow[u, dashed, "\tilde{\varphi}"'] \end{tikzcd}.$$

(b) Laurent: $X(f, g^{-1}) \subseteq X$, $f = (f_1, \dots, f_r)$, $g = (g_1, \dots, g_s^{-1})$.

Step 1 Construct $A\langle f, g^{-1}\rangle = A\langle \zeta_1, \dots, \zeta_r, \xi_1, \dots, \xi_s\rangle / (\boxed{\zeta_i - f_i}, \boxed{\xi_j g_j - 1})$.

Step 2 Check $\operatorname{im} \iota \subseteq X(f, g^{-1})$ for $\iota : \operatorname{Sp} A\langle f, g^{-1}\rangle \to X = \operatorname{Sp} A$.

$\zeta_i - \iota^*(f_i) = 0$, $|\zeta_i|_{\sup} \le 1$

$\iota^*(g_j)\xi_j = 1$, $|\xi_j|_{\sup} \le 1$.

Step 3 Check univ property.

(c) Rational: $X(\frac{f}{f_0}) \subseteq X$, $\frac{f}{f_0} = (\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0})$.

$\rightsquigarrow$ Construct $A\langle \frac{f}{f_0}\rangle = A\langle \zeta_1, \dots, \zeta_r\rangle / (f_1 - f_0\zeta_1, \dots, f_r - f_0\zeta_r)$. $\square$

Prop (Transitivity of aff subdoms)

$U \subset V \subset X = \operatorname{Sp} A$ aff subdoms $\Rightarrow$ $U \subseteq X$ aff subdom.

Prop $\varphi : Y \to X$ of aff spaces. $X' \subseteq X$ aff subdom.

(1) $\exists!\ \varphi' : Y' \to X'$ s.t.
$$\begin{tikzcd} Y' \arrow[r, "\varphi'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$
Cartesian.

i.e. $Y' = \varphi^{-1}(X') = Y \times_X X'$.

(2) $\varphi^{-1}(X(f)) = Y(\varphi^* f)$,

$\varphi^{-1}(X(f, g^{-1})) = Y(\varphi^* f, (\varphi^* g)^{-1})$,

$\varphi^{-1}(X(\frac{f}{f_0})) = Y(\frac{\varphi^* f}{\varphi^* f_0})$, $(f) = A$.

Namely, morphisms preserve special aff subdoms.

Proof (1)
$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!"'] \arrow[drr, dashed, bend left, "\exists!"] \arrow[ddr, bend right, "\psi"'] & & \\ & Y' \arrow[r, "\varphi'"] \arrow[d] & X' \arrow[d] \\ & Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

Take $\psi : Z \to Y$ s.t. $\operatorname{im} \psi \subseteq Y'$

$\Rightarrow \operatorname{im}(\varphi \circ \psi) \subseteq \varphi(Y') \subseteq X'$.

$\Rightarrow \exists!\ Z \to X'$ by univ property

Take $Y' = \varphi^{-1}(X') \Rightarrow \exists!\, Z \to Y'$ s.t. $\psi : Z \to Y' \hookrightarrow Y$.

$\Rightarrow \quad Y' \simeq Y \times_X X' \subseteq Y$ aff subdom.

Note that (1) $\Rightarrow$ (2) as $Y \times_X X(f) = \varphi^{-1}(X(f)) = Y(\varphi^* f)$, etc.. $\square$

<u>Prop</u> $U, V \subseteq X$ general (resp. Weierstrass/Laurent/rational) aff subdoms. Then so also is $U \cap V$. In particular,

$$\left\{\begin{matrix}\text{Weierstrass doms} \\ X(f)\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{Laurent doms} \\ X(f, g^{-1})\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{rational doms} \\ X(\frac{f}{f_0})\end{matrix}\right\}$$

trivial by Prop. $X(f, g^{-1}) = X(f) \cap X(\frac{1}{g})$

<u>Proof</u>

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, hook] \\ U \arrow[r] & X \end{tikzcd} \quad \Rightarrow \quad U \cap V \subseteq U \text{ aff subdom.}$$

If $U = X(\frac{f}{f_0})$, $V = X(\frac{g}{g_0})$ rational subdoms, then

(1) $(f, f_0) = (g, g_0) = A \Rightarrow (f_i g_j)_{\substack{0 \le i \le r \\ 0 \le j \le s}} = A$

(2) $\forall x \in U \cap V$, $|(f_i g_j)(x)| \le |(f_0 g_0)(x)|$

$\Rightarrow |(f_i g_0)(x)| \le |(f_0 g_0)(x)| \Rightarrow |f_i(x)| \le |f_0(x)|$.

So $U \cap V = X(\frac{f_i g_j}{f_0 g_0})_{i,j \ne 0}$. $\square$

<u>Thm</u> (Gerritzen-Grauert) $U \subseteq X = \operatorname{Sp} A$ aff subdom.

$\Rightarrow \quad U = \bigcup_{i=1}^{n} X(\frac{f_i}{f_{0i}})$, $\quad f_i = (f_{i1}, \cdots, f_{ik(i)})$.

In particular, any aff subdom is canonically open.