

# Affinoid subdomains.

Jan 12

## §1 Affinoid spaces (recap)

$K$  non-arch local field.

Take alg  $T_n = K\langle S_1, \dots, S_n \rangle \cong \mathcal{O}$  ideal.

$\mapsto T_n/\mathcal{O} = A$  affinoid algebra.

Def'n (1) The affinoid  $K$ -space is the datum

$$\mathrm{Sp} A := (\mathrm{Max} A, A)$$

Note Algebraic geom:  $\mathrm{Spec} A \leftrightarrow$  prime ideals

Rigid geom:  $\mathrm{Sp} A (= \mathrm{Spm} A) \leftrightarrow$  max'l ideals

(2) The vanishing locus of  $\mathcal{O}$  is

$$\begin{aligned} V(\mathcal{O}) &:= \{x \in \mathrm{Sp} A : f(x) = 0, \forall f \in \mathcal{O}\} \\ &= \{x \in \mathrm{Sp} A : \mathcal{O} \subseteq x\}. \end{aligned}$$

(3) The basic open subset is

$$D_f := \{x \in \mathrm{Sp} A : f(x) \neq 0\}, \quad f \in A.$$

Prob How about  $V(\mathcal{O}) = \{x \in \mathbb{B}^n(\bar{K}) : \forall f \in \mathcal{O}, f(x) = 0\}$ ?

•  $f \in A \mapsto$  function  $\mathrm{Sp} A \rightarrow \bar{K}$ .

•  $f \in T_n \mapsto$  function  $\mathbb{B}^n(\bar{K}) \rightarrow \bar{K}$ .

$$\mathbb{B}^n(\bar{K})/\mathrm{Gal}(\bar{K}/K)$$

$$\mapsto \mathrm{Max} T_n \rightarrow \mathrm{Max} A$$

Prop (1)  $\mathcal{O} \subseteq \mathcal{b} \Rightarrow V(\mathcal{O}) \supseteq V(\mathcal{b})$ ;  $V(\sum \mathcal{O}_i) = \bigcap V(\mathcal{O}_i)$ ;  $V(\mathcal{O}\mathcal{b}) = V(\mathcal{O}) \cup V(\mathcal{b})$ .

(2)  $\{D_f\}_{f \in A}$ : Zariski top basis on  $\mathrm{Sp} A$ .

(3) (Hilbert Nullstellensatz).

$Y \subseteq \mathrm{Sp} A$ , define  $\mathrm{id}(Y) := \bigcap_{y \in Y} \mathfrak{m}_y \subseteq A = \{f \in A : f(y) = 0, \forall y \in Y\}$ .

Then  $V(\mathrm{id}(Y)) = \bar{Y}$ ,  $\mathrm{id}(V(\mathcal{O})) = \sqrt{\mathcal{O}}$ .

Cor  $f_1, \dots, f_n \in A$ . Then  $(f_1, \dots, f_n) = A \iff \cup D_{f_i} = \text{Sp } A$ .

Thm There's an equivalence of cats:

$$\{\text{affinoid } K\text{-algebras}\}^{\text{opp}} \simeq \{\text{affinoid } K\text{-spaces}\}$$

$$A \longleftrightarrow \text{Sp } A.$$

E.g. (1)  $\sigma: A \longrightarrow B \rightsquigarrow \text{Sp } B \longrightarrow \text{Sp } A$ .

still max'l  $\rightarrow \boxed{\sigma^{-1}(m)}$   $\begin{matrix} \cup \\ m \end{matrix} \quad \begin{matrix} \cup \\ m \end{matrix} \rightsquigarrow \sigma^{-1}(m)$ .

Also,  $K \hookrightarrow A/\sigma^{-1}(m) \hookrightarrow B/m$  monomorphisms.

(2) Functoriality: e.g.

$$\begin{array}{ccc} R \longrightarrow S & & \text{Sp } R \longleftarrow \text{Sp } S \\ \downarrow \quad \downarrow & \longleftrightarrow & \uparrow \quad \uparrow \\ T \longrightarrow \widehat{S \otimes_R T} & & \text{Sp } T \longleftarrow \text{Sp}(S \widehat{\otimes}_R T) \\ & & \text{Sp } S \cong_{\text{Sp } R} \text{Sp } T. \end{array}$$

## §2 Canonical topology

Motivation Rigid space:  $X = \text{Sp } A$  + Grothendieck top  $\left\{ \begin{array}{l} \text{adm opens} \\ \text{adm open coverings} \end{array} \right.$

$\rightsquigarrow$  sheaf  $\mathcal{O}_X$  of rigid analytic functions

$$\text{s.t. } A = \Gamma(X, \mathcal{O}_X).$$

$\rightsquigarrow$  non-arch analogue  $\mathbb{C}_p \cong \mathbb{C}$  of alg geom/ $\mathbb{C}$  or arith geom/ $\mathbb{Q}$

canonical top / Zar top      Zar top      conf top / Zar top.

& non-arch norms      & disc.

e.g. rigid-analytic GAGA, coh comparison, uniformizations of Shimura varieties, etc.

Example  $X = \text{Sp } T_1 = \text{Sp } K\langle \varpi \rangle$  closed unit disc.

$Y = U \sqcup S$ ,  $U =$  open unit disc,  $S = \text{Sp } k\langle \xi, \xi^{-1} \rangle$  boundary.  
 $\hookrightarrow Y \rightarrow X$  open bijective immersion, but  $Y \neq X$  as aff spaces.

b/c  $U \neq \bigcup_{\text{aff}} \text{subdomains}$

$\Rightarrow \{U, S\} \neq$  adm open cover under Grothendieck top.

as a set  $\xrightarrow{(\text{Max } T_n, T_n)}$

Fix  $T_n \rightarrow A \hookrightarrow \text{Sp } A \hookrightarrow \text{Sp } T_n = \text{Max } T_n \longleftarrow \mathbb{B}^n(\mathbb{K})$   
 as a top space

$A_{\mathbb{K}}\text{-top} \hookrightarrow \mathbb{B}^n(\mathbb{K})\text{-top} \hookrightarrow \text{Sp } T_n\text{-top} \hookrightarrow \text{Sp } A\text{-top}$

- indep't of the Zariski top & the choice of  $\alpha: T_n \rightarrow A$ .
- inherited from  $\mathbb{K}$ -top (intrinsic).

For  $\varepsilon > 0$ ,  $f \in A$ ,  $X = \text{Sp } A$ , define

$$X(f; \varepsilon) := \{x \in X : |f(x)| = \varepsilon\},$$

$$X(f) := X(f, 1),$$

$$X(f_1, \dots, f_r) := X(f_1) \cap \dots \cap X(f_r).$$

Def'n A canonical top on  $\text{Sp } A$  is the top gen'd by  $\{X(f; \varepsilon)\}_{f \in A, \varepsilon > 0}$  as open subsets.

Prop  $X = \text{Sp } A$ , can top on  $X$  is gen'd by  $\{X(f)\}_{f \in A}$ .

i.e.  $\forall U \subseteq X$  open,  $U = \bigcup U_i$ ,  $U_i = X(f_{i1}, \dots, f_{in(i)})$ .

Proof  $X(f; \varepsilon) = \bigcup_{\substack{\varepsilon' = \varepsilon \\ \varepsilon' \in \mathbb{K}^{\times}}} X(f; \varepsilon')$

$x \in \text{Max } A, (A/x)/\mathbb{K}$ .  $f \mapsto f(x)$   
 $(|f(x)| \in \mathbb{K}^{\times}, \forall x \in \mathbb{K})$ .

$\forall \varepsilon' \in \mathbb{K}^{\times}$ ,  $\exists \delta$  s.t.  $(\varepsilon')^s \in \mathbb{K}^{\times}$  by def'n.

$$\Rightarrow X(f; \varepsilon) = X(f^s; (\varepsilon')^s) \supseteq X(c^t f^s; 1) = X(c^t f^s).$$

□

$$|f(x)^s| = |f(x)|^s \leq (\varepsilon')^s = |c| \Leftrightarrow |c^t f(x)^s| = \frac{1}{c} |f(x)|^s \leq 1.$$

LEM Let  $X = \text{Sp} A \ni x$ ,  $f \in A$ ,  $|f(x)| = \varepsilon > 0$ .

Then  $\exists g \in A$  s.t.  $g(x) = 0$ , and  $(y \in X(g) \Rightarrow |f(y)| = \varepsilon)$ .

In particular,  $X(g) \subseteq \{y \in X : |f(y)| = \varepsilon\}$  is an open nbhd of  $x$ .

Proof Consider  $A \rightarrow \boxed{A/m_x} \leftarrow /K$   
 $f \mapsto \bar{f} = f(x)$ .

$$\hookrightarrow P(\zeta) = \zeta^n + c_1 \zeta^{n-1} + \dots + c_n \in K[\zeta] \text{ min poly of } f(x) \in A/m_x \text{ over } K.$$

$$= \prod_{i=1}^n (\zeta - \alpha_i), \quad \alpha_i \in \bar{K}$$

$$\Rightarrow \forall 1 \leq i \leq n, \quad \varepsilon = |f(x)| = |\alpha_i| \quad \text{by def'n of 1:1 on } A/m_x.$$

Take  $g = P(f) \in A$ .

$$\text{Check: (i) } g(x) = P(f(x)) = P(\bar{f}) = 0.$$

$$(ii) \forall y \in X, |g(y)| \leq 1 \stackrel{?}{\Rightarrow} |f(y)| = \varepsilon$$

Claim  $|g(y)| < \varepsilon^n \Rightarrow |f(y)| = \varepsilon$ .

$$\text{Otherwise, } |f(y) - \alpha_i| \stackrel{?}{=} \max(|f(y)|, |\alpha_i|) \geq \varepsilon.$$

$$\uparrow$$

$$|f(y)| \neq |\alpha_i|$$

But  $|g(y)| = |P(f(y))| = \prod |f(y) - \alpha_i| \geq \varepsilon^n$ , contradiction.

So  $\exists c \in K^*$ ,  $|c| < \varepsilon^n$ . s.t.  $X(c \cdot g) \subseteq \{y \in X : |f(y)| = \varepsilon\}$ .

(May replace  $g$  with  $c \cdot g$  if necessary).  $\square$

COR (i) The following are open w.r.t. can top:

$$\{x \in \text{Sp} A : |f(x)| = \varepsilon \text{ (resp. } \leq \varepsilon, \geq \varepsilon)\}.$$

$$\{x \in \text{Sp} A : f(x) \neq 0\} = \{x \in \text{Sp} A : |f(x)| \neq 0\}.$$

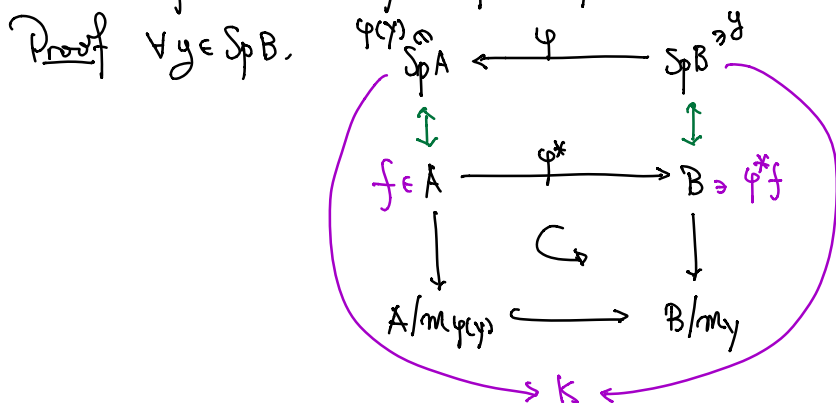
(ii)  $X = \text{Sp} A$ ,  $\forall x \in X$ ,  $X(f_1, \dots, f_r)$ 's with  $f_1, \dots, f_r \in m_x$   
 form a basis of nbhd of  $x$ .



Prop  $\varphi: \text{Sp} B \rightarrow \text{Sp} A$ ,  $\varphi^*: A \rightarrow B$  morphisms,  $f_1, \dots, f_r \in A$ .

Then  $\varphi^{-1}(\underbrace{(\text{Sp} A)(f_1, \dots, f_r)}_{\text{open of Sp} A}) = \underbrace{\text{Sp}(B)(\varphi^*(f_1), \dots, \varphi^*(f_r))}_{\text{open of Sp} B}$ .

In particular, any morphism  $\varphi$  is continuous.



$$\Rightarrow |f(\varphi(y))| = |(\varphi^* f)(y)|, \quad \forall f \in A$$

$$\Rightarrow \varphi^{-1}((\text{Sp} A)(f)) = (\text{Sp} B)(\varphi^*(f)). \quad \square$$

### §3 Affinoid subdomains

Def'n (Spatial aff subdomains)  $X = \text{Sp} A$ .

(1) Weierstrass dom:  $X(f_1, \dots, f_r) = \{x \in X : |f_i(x)| \leq 1\}$

(2) Laurent dom:  $X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{x \in X : |f_i(x)| \leq 1, |g_j(x)| \geq 1\}$ .

(3) Rational dom:  $X(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0}) = \{x \in X : |f_i(x)| \leq |f_0(x)|\}$

where  $f_0, \dots, f_r$  has no common zeros, i.e.  $(f_0, \dots, f_r) = A$ .

Bank In (3),  $(f_0, \dots, f_r) = A$  cannot be dropped.

E.g.  $X = \text{Sp} T_1 = \text{Sp} K\langle S \rangle$ ,  $|c| \cdot |S(x)|$

$$\rightsquigarrow X(\frac{S}{cS}) = \{x \in X : |S(x)| \leq |cS(x)| \text{ for } c \in K, 0 < |c| < 1\}$$

$$= \{x \in X : |S(x)| = 0\}$$

= pt, cannot be open.

lem Special doms are open w.r.t. can top.

Proof Weierstrass/Laurent: Done.

Rational:  $\forall x \in X(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0})$ ,  $|f_0(x)| \neq 0$ ,

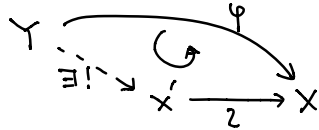
$\exists$  open nbhd  $U = X(f_1; f_1(x)) \cap \dots \cap X(f_r; f_r(x)) \cap \{y: |f_0(y)| \geq |f_0(x)|\}$ .  
(finite intersection of opens)

Check:  $x \in U$  obvious.  $\forall y \in U$ ,  $|f_i(y)| \leq |f_i(x)|$ ,  $|f_0(y)| \geq |f_0(x)|$   
 $\Rightarrow |f_i(y)| \leq |f_0(y)| \Rightarrow y \in X(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0})$ .  $\square$

Def:  $X = \text{Sp} A$ . A subset  $U \subseteq X$  is an affinoid subdomain if

(i)  $\exists \zeta: X' \rightarrow X$  s.t.  $\text{im } \zeta \subseteq U$ ,

(ii)  $\forall \varphi: Y \rightarrow X$  s.t.  $\varphi(Y) \subseteq U$ ,  $\exists! Y \xrightarrow{f} X'$  s.t.



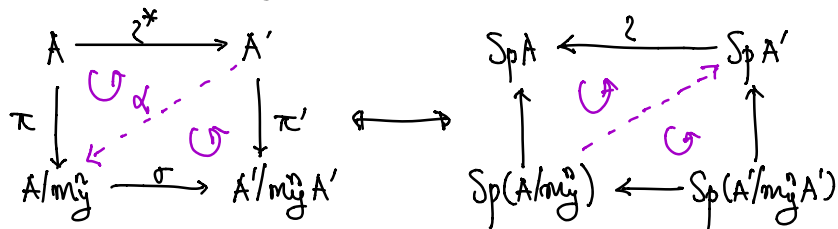
lem  $X = \text{Sp} A$ ,  $X' = \text{Sp} A'$ ,  $U \subseteq X$  aff subdom } as above  
 $(\zeta, \zeta^*): X' \rightarrow X$ ,  $\zeta^*: A \rightarrow A'$ .

Then (1)  $\zeta$  injective,  $\zeta(X') = U$  ( $\hookrightarrow$  bijection  $X' \xrightarrow{\sim} U$ ).

(2)  $\forall x \in X'$ ,  $n \in \mathbb{N}$ ,  $A/m_{\zeta(x)}^n \xrightarrow{\sim} A'/m_x^n$ .

(3)  $\forall x \in X'$ ,  $m_x = m_{\zeta(x)} A'$ .

Proof  $\forall y \in U$ ,



Notice that  $\text{Max } A/m_y^n = \{m_y\} = \text{Sp } A/m_y^n$ .

$\Rightarrow \text{im}(\text{Sp}(A/m_y^n) \rightarrow \text{Sp} A) = \{y\} \subseteq U$ .

$$\Rightarrow \exists! \text{Sp} A / \mathfrak{m}_y \rightarrow \text{Sp} A' \quad \text{by univ property}$$

$$\Rightarrow \exists! \alpha: A' \rightarrow A / \mathfrak{m}_y \text{ s.t. } \pi = \alpha \circ \zeta^*, \pi' = \sigma \circ \alpha.$$

Now  $\pi$  surj  $\Rightarrow \alpha$  surj,  $\pi'$  surj  $\Rightarrow \sigma$  surj.

Have  $\ker \pi' = \mathfrak{m}_y A' \subset \ker \alpha \Rightarrow \ker \sigma = 0 \Rightarrow \sigma$  inj.

For  $n=1$ ,  $\mathfrak{m}_{2(x)} A' = \mathfrak{m}_x \Rightarrow (1)(3)$ .

And  $\sigma$  bijective  $\Rightarrow (2)$ .  $\square$

Prop  $X = \text{Sp} A$ . Special aff subdoms are truly aff subdoms.

lem (For checking  $\text{im} \zeta \subseteq U$ ).

Given  $\varphi: Y \rightarrow X$ ,  $\forall y \in Y$ ,  $f_1, \dots, f_r \in A$ ,

$$\text{we have } \varphi(Y) \subseteq X(f_1, \dots, f_r) \iff \forall i, |f_i(\varphi(y))| \leq 1$$

$$|(\varphi^* f_i)(y)|$$

$$\text{(by max principle)} \iff \|\varphi^* f_i\|_{\text{sup}} \leq 1, \forall i.$$

Proof (a) Weierstrass:  $f := (f_1, \dots, f_r) \mapsto X(f) \subseteq X$ .

$$\text{Let } A\langle f \rangle = A\langle f_1, \dots, f_r \rangle = A\langle \delta_1, \dots, \delta_r \rangle / (\delta_1 - f_1, \dots, \delta_r - f_r).$$

$$\hookrightarrow \zeta^*: A \rightarrow A\langle f \rangle, \quad \zeta: \text{Sp} A\langle f \rangle \rightarrow X.$$

Check (1)  $\text{im} \zeta \subseteq \text{Sp} X(f)$ : for some fixed  $\alpha: T_n \rightarrow A$ ,

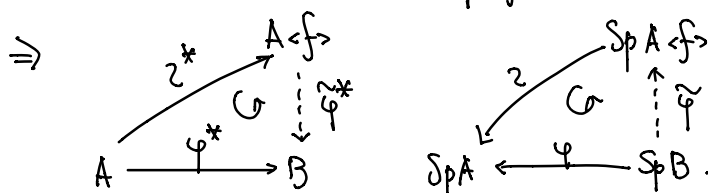
$$\|\zeta^*(f_i)\|_{\text{sup}} \leq \|\delta_i\|_{\text{sup}} \leq \|\delta_i\|_{\alpha} \leq 1.$$

(2) Univ property:

Assume  $\varphi: Y = \text{Sp} B \rightarrow X$  s.t.  $\varphi(Y) \subseteq X(f) \mapsto \varphi^*: A \rightarrow B$

$\hookrightarrow$  extension  $\tilde{\varphi}^*: A\langle f \rangle \rightarrow B \Rightarrow \delta_i - f_i \in \ker \tilde{\varphi}^*$ .

$$\delta_i \longmapsto \varphi^*(f_i)$$



(b) Laurent:  $X(f, g^{-1}) \subseteq X$ ,  $f = (f_1, \dots, f_r)$ ,  $g = (g_1, \dots, g_s^{-1})$ .

Step 1 Construct  $A\langle f, g^{-1} \rangle = A\langle \xi_1, \dots, \xi_r, \xi_{r+1}, \dots, \xi_{r+s} \rangle / (\xi_i - f_i, \xi_j g_j^{-1})$ .

Step 2 Check  $\text{im } \iota \subseteq X(f, g^{-1})$  for  $\iota: \text{Sp } A\langle f, g^{-1} \rangle \rightarrow X = \text{Sp } A$ .

$$\xi_i - \iota^*(f_i) = 0, \quad |\xi_i|_{\text{sup}} = 1$$

$$\iota^*(g_j) \xi_j = 1, \quad |\xi_j|_{\text{sup}} = 1.$$

Step 3 Check univ property.

(c) Rational:  $X(\frac{f}{f_0}) \subseteq X$ ,  $\frac{f}{f_0} = (\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0})$ .

$\hookrightarrow$  Construct  $A\langle \frac{f}{f_0} \rangle = A\langle \xi_1, \dots, \xi_r \rangle / (f_i - f_0 \xi_i, \dots, f_r - f_0 \xi_r)$ .  $\square$

Prop (Transitivity of aff subdoms)

$U \subset V \subset X = \text{Sp } A$  aff subdoms  $\Rightarrow U \subseteq X$  aff subdom.

Prop  $\varphi: Y \rightarrow X$  of aff spaces,  $X' \subseteq X$  aff subdom.

$$(1) \exists! \varphi': Y' \rightarrow X' \text{ s.t. } \begin{array}{ccc} Y' & \xrightarrow{\varphi'} & X' \\ \downarrow \Gamma & & \downarrow \\ Y & \xrightarrow{\varphi} & X \end{array} \text{ Cartesian.}$$

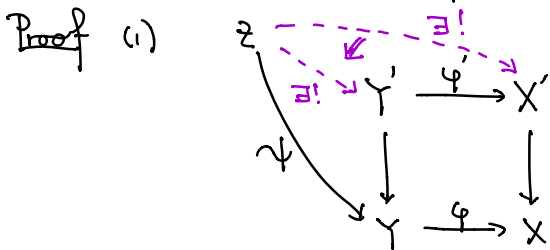
$$\text{i.e. } Y' = \varphi'^{-1}(X') = Y \times_X X'.$$

$$(2) \varphi^{-1}(X(f)) = Y(\varphi^* f).$$

$$\varphi^{-1}(X(f, g^{-1})) = Y(\varphi^* f, (\varphi^* g)^{-1}),$$

$$\varphi^{-1}(X(\frac{f}{f_0})) = Y(\frac{\varphi^* f}{\varphi^* f_0}), \quad (f) = A.$$

Namely, morphisms  
preserve special aff subdoms.



Take  $\psi: Z \rightarrow Y$  s.t.  $\text{im } \psi = Y'$

$$\Rightarrow \text{im}(\varphi \circ \psi) \subseteq \varphi(Y') \subseteq X'$$

$\Rightarrow \exists! z \rightarrow X'$  by univ property

Take  $Y' = \varphi^{-1}(X') \Rightarrow \exists! Z \rightarrow Y'$  s.t.  $\psi: Z \rightarrow Y' \hookrightarrow Y$ .

$$\Rightarrow Y' \cong Y \times_X X' \subseteq Y \text{ aff subdom.}$$

Note that (1)  $\Rightarrow$  (2) as  $Y \times_X X(f) = \varphi^{-1}(X(f)) = Y(\varphi^*f)$ , etc..  $\square$

Prop  $U, V \subseteq X$  general (resp. Weierstrass/Laurent/rational) aff subdoms.

Then so also is  $U \cap V$ . In particular,

$$\left\{ \begin{array}{l} \text{Weierstrass doms} \\ X(f) \end{array} \right\} \begin{array}{c} \supseteq \\ \uparrow \\ \text{trivial} \end{array} \left\{ \begin{array}{l} \text{Laurent doms} \\ X(f, g^{-1}) \end{array} \right\} \begin{array}{c} \supseteq \\ \uparrow \\ \text{by Prop. } X(f, g^{-1}) = X(f) \cap X(\frac{1}{g}) \end{array} \left\{ \begin{array}{l} \text{rational doms} \\ X(\frac{f}{g}) \end{array} \right\}$$

Proof

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow \ulcorner & & \downarrow \\ U & \longrightarrow & X \end{array} \Rightarrow U \cap V \subseteq U \text{ aff subdom.}$$

If  $U = X(\frac{f}{f_0})$ ,  $V = X(\frac{g}{g_0})$  rational subdoms. then

$$(1) (f, f_0) = (g, g_0) = A \Rightarrow (f_i g_j)_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s}} = A$$

$$(2) \forall x \in U \cap V, |(f_i g_j)(x)| \leq |(f_0 g_0)(x)|$$

$$\Rightarrow |(f_i g_0)(x)| \leq |(f_0 g_0)(x)| \Rightarrow |f_i(x)| \leq |f_0(x)|.$$

$$\text{So } U \cap V = X\left(\frac{f_i g_i}{f_0 g_0}\right)_{i,j \neq 0}.$$

$\square$

Then (Gerritzen-Gravert)  $U \subseteq X = \text{Sp}A$  aff subdom.

$$\Rightarrow U = \bigcup_{i=1}^n X\left(\frac{f_i}{f_{0i}}\right), \quad f_i = (f_{i1}, \dots, f_{ik(i)})$$

In particular, any aff subdom is canonically open.