2023 Winter – Seminar on *Formal and Rigid Geometry*

Talk 5

AFFINOID FUNCTIONS (I)

This is the live-TeXed notes by Wenhan Dai for this seminar. The note-taker claims no originality and takes full responsibility for all errors made therein.

- **Talk 5 (Affinoid functions I):** Cover [Bos14, pp.65–76]. Describe presheaf $\mathcal{O}_X$ and stalks $\mathcal{O}_{X,x}$. Prove §4.1 Proposition 2 and Proposition 6. Then discuss locally closed immersion and Runge immersion. Finally, state extension Lemma 9 for the preparation of next talk.
- **Talk 6 (Affinoid functions II):** Cover [Bos14, pp.76–91]. Understand Gerritzen-Grauert theorem (§4.2, Theorem 10) and sketch the proof. Understand the Tate's acyclic theorem (§4.3, Theorem 1) and sketch the proof. If time permits, discuss the generalized version of Tate's acyclic theorem (Theorem 10, Corollary 11) (For more details, see [BRG84, Chap.8]).

Readme. Talk 6 follows the book [Bos14] very closely so we only build the notes for Talk 5.

Recap. Before talking about affinoid functions we first recall some basic notions. Let K be a complete non-archimedean valuation field. The Tate algebra over K is $T_n = K\langle\zeta_1, \dots, \zeta_n\rangle$ and the affinoid K-algebra A is defined as the datum of a quotient of T_n together with a morphism $T_n \twoheadrightarrow A$ in the category of affinoid K-algebras. We have also defined (c.f. Talk 3–4) affinoid K-spaces via $X = \mathrm{Sp}(A)$, which is set-theoretically the collection of maximal ideals of A.

1. Germs of affinoid functions

Let X be an affinoid K-space and $U \subset X$ an affinoid subdomain. We are to consider the functor

$$\begin{aligned} \mathcal{O}_X : \{U \subset X \text{ affinoid subdomains}\} &\longrightarrow \mathsf{Ring} \\ U &\longmapsto \mathcal{O}_X(U). \end{aligned}$$

This $\mathcal{O}_X(-)$ gives a presheaf of affinoid functions on X, which is compatible with subset restrictions: for $V \subset U$ we can take $f \mapsto f|_V$ naturally. This $\mathcal{O}_X$ is called the *presheaf of affinoid functions* on X. For $x \in X$, we define the *stalk* at x by

$$\mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U),$$

where the injective limit runs over all affinoid subdomains U of X. An element $f_X \in \mathcal{O}_{X,x}$ is called a *germ* of affinoid functions at x. As one would expect, given a morphism $\varphi : X \to Y$ of affinoid spaces, there exists a natural induced morphism of stalks for each $y \in Y$, read as

$$\varphi_y : \mathcal{O}_{X,\varphi(y)} \longrightarrow \mathcal{O}_{Y,y}.$$

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Proposition 1.1 (Localization compatibility)**.** *Let X be an affinoid K-space, and $x \in X$, a point corresponding to some maximal ideal $\mathfrak{m} \subset \mathcal{O}_X(X)$. Then $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}\mathcal{O}_{X,x}$.*

Proof. Consider $U \ni x$ an affinoid subdomain of X. Then by Talk 3–4, $\mathfrak{m}\mathcal{O}_X(U)$ is a maximal ideal of $\mathcal{O}_X(U)$, and

$$\mathcal{O}_X(X)/\mathfrak{m} \xrightarrow{\sim} \mathcal{O}_X(U)/\mathfrak{m}\mathcal{O}_X(U).$$

We obtain an exact sequence

$$0 \to \mathfrak{m}\mathcal{O}_X(U) \to \mathcal{O}_X(U) \to \mathcal{O}_X(U)/\mathfrak{m}\mathcal{O}_X(U) \to 0.$$

In fact, for an exact sequence of inductive systems, taking injective limits preserve exactness. So, on the level of stalks,

$$0 \to \mathfrak{m}\mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \to \mathcal{O}_X(X)/\mathfrak{m} \to 0.$$

Then we see $\mathfrak{m}\mathcal{O}_{X,x}$ is a maximal ideal of $\mathcal{O}_{X,x}$.

Now let $f_x \in \mathcal{O}_{X,x}$, then whenever $f_x \notin \mathfrak{m}\mathcal{O}_{X,x}$, we expect f_x to be a unit in $\mathcal{O}_{X,x}$. To show this, we assume f_x is represented by $f \in \mathcal{O}_X(U)$. Then $f_x \notin \mathfrak{m}\mathcal{O}_{X,x}$. Thus, $f \notin \mathfrak{m}\mathcal{O}_X(U)$, and hence $f(x) \neq 0$. After multiplying a constant $c \in K^*$ if necessary, we may assume $|f(x)| \geqslant 1$. Then $x \in U(f^{-1})$, a Laurent subdomain of U, and hence an affinoid subdomain of X, in which $f|_{U(f^{-1})}$ is invertible. So f_x must be invertible. □

Proposition 1.2. *Let $X = \operatorname{Sp} A$ be an affinoid K-space, and $x \in X$ corresponding to some maximal ideal $\mathfrak{m}$ of A. Then the canonical map $A \to \mathcal{O}_{X,x}$ of affinoid K-algebras decomposes as*

$$A \longrightarrow A_{\mathfrak{m}} \longrightarrow \mathcal{O}_{X,x},$$

where the first map is the canonical map of localization at $\mathfrak{m}$, and the second map is injective. Furthermore, these two maps induce isomorphisms

$$A/\mathfrak{m}^n \xrightarrow{\sim} A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}, \quad \forall n \in \mathbb{N}.$$

So, via taking projective limits of $\mathfrak{m}$-adic completions, we obtain isomorphisms

$$\widehat{A} \xrightarrow{\sim} \widehat{A_{\mathfrak{m}}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}.$$

Proof. The decomposition comes from that $A/\mathfrak{m} \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x}$, and $\mathcal{O}_{X,x}$ is local. Fox $x \in U \subset X$ an affinoid subdomain, let $A' = \mathcal{O}_X(U)$. Then there exists an isomorphism $A/\mathfrak{m}^n \xrightarrow{\sim} A'/\mathfrak{m}^n A'$. Consider the exact sequence of inductive systems

$$0 \to \{\mathfrak{m}^n A'\} \to \{A\} \to \{A'/\mathfrak{m}^n A'\} \to 0.$$

Passing to injective limits, we have an exact sequence

$$0 \to \mathfrak{m}^n\mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \to A/\mathfrak{m}^n \to 0.$$

So we see that the natural map $A/\mathfrak{m}^n \to \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}$ is an isomorphism. Consequently,

$$A/\mathfrak{m}^n \xrightarrow{\sim} A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}$$

are isomorphisms. Dually, by passing to projective limits, we have

$$\widehat{A} \xrightarrow{\sim} \widehat{A_{\mathfrak{m}}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}.$$

Finally, to show $A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$ is injective, we consider the commutative diagram

$$\begin{array}{ccc} A_{\mathfrak{m}} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \widehat{A_{\mathfrak{m}}} & \xrightarrow{\sim} & \widehat{\mathcal{O}}_{X,x}. \end{array}$$

Since $A_{\mathfrak{m}}$ is a noetherian local ring, by Krull's intersection theorem, $\bigcap_n \mathfrak{m}^n A_{\mathfrak{m}} = 0$, and hence $A_{\mathfrak{m}} \to \widehat{A_{\mathfrak{m}}}$ is injective. It follows that $A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$ is injective. □

We want to derive some direct consequence of the injectivity of map $A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$ in Proposition 1.2.

Corollary 1.3. *An affinoid function f on some affinoid K-space X is trivial if and only if its germ $f_x \in \mathcal{O}_{X,x}$ is trivial at any $x \in X$.*

Proof. Write $X = \operatorname{Sp} A$. The assertion is clear via

$$A \hookrightarrow \prod_{\mathfrak{m} \in X} A_{\mathfrak{m}} \hookrightarrow \prod_{x \in X} \mathcal{O}_{X,x}.$$

□

Corollary 1.4. *Let X be an affinoid K-space equipped with a covering $X = \bigcup_{i \in I} U_i$ by affinoid subdomains. Then the restriction maps $\mathcal{O}_X(X) \to \mathcal{O}_X(U_i)$ define an injection*

$$\mathcal{O}_X(X) \hookrightarrow \prod_{i \in I} \mathcal{O}_X(U_i).$$

Corollary 1.5. *For any affinoid subdomain $X' = \operatorname{Sp} A'$ of some affinoid K-space $X = \operatorname{Sp} A$, the map $A \to A'$ of K-algebras is flat.*

Proof. It suffices to show for any $x \in X'$ corresponding to $\mathfrak{m} \in \operatorname{Max} A$, the map

$$f : A_{\mathfrak{m}} \to A'_{\mathfrak{m}'}$$

is flat, where $\mathfrak{m}' = \mathfrak{m}A'$. Consider the commutative diagram

$$\begin{array}{ccc} A_{\mathfrak{m}} & \xrightarrow{f} & A'_{\mathfrak{m}} \\ \downarrow & & \downarrow g \\ \widehat{A_{\mathfrak{m}}} & \longrightarrow & \widehat{A'}_{\mathfrak{m}'} \end{array}$$

where the vertical maps are injective and faithfully flat. By Proposition 1.2, we obtain an isomorphism $\widehat{A_{\mathfrak{m}}} \xrightarrow{\sim} \widehat{A'}_{\mathfrak{m}'}$. So the composition $g \circ f$ is flat. However, g itself is faithfully flat, and hence f must be flat. □

Proposition 1.6. *Let X be an affinoid K-space. For any point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is noetherian.*

Proof. Let $X = \operatorname{Sp} A$ and $\mathfrak{m}$ be the maximal ideal of A corresponding to x. Then we claim that $\mathcal{O}_{X,x}$ is $\mathfrak{m}$-adically separated, i.e.

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n \mathcal{O}_{X,x} = 0.$$

To prove this, take $f_x \in \bigcap_{n=1}^{\infty} \mathfrak{m}^n \mathcal{O}_{X,x}$. Then f_x is represented by $f \in \mathcal{O}_X(U)$ for some $x \in U \subset X$, where U is an affinoid subdomain of X. From Proposition 1.2, we have an isomorphism

$$\mathcal{O}_X(U)/\mathfrak{m}^n \mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}^n \mathcal{O}_{X,x}.$$

Therefore, we see $f \in \bigcap_{n=1}^{\infty} \mathfrak{m}^n \mathcal{O}_X(U)$. Passing to the level of local rings, we deduce $f_x = 0$.

In the same way, one can show for any finitely generated ideal $\mathfrak{a}_x \subset \mathcal{O}_{X,x}$ that the residue ring $\mathcal{O}_{X,x}/\mathfrak{a}_x$ is $\mathfrak{m}$-adically separated. Since $\mathfrak{a}_x$ is finitely generated, one may assume the generators of $\mathfrak{a}_x$ are represented by affinoid functions $f_i \in \mathcal{O}_X(U)$ for some affinoid subdomain $x \in U \subset X$. Now replacing X by U, we may assume $f_i \in A \simeq \mathcal{O}_X(X)$ and $\mathfrak{a}_x$ is induced from some ideal $\mathfrak{a} \subset A$. Then turn to consider $Y = \operatorname{Sp}(A/\mathfrak{a})$, with $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}/\mathfrak{a}_x$. Thus, $\mathcal{O}_{X,x}/\mathfrak{a}_x$ is $\mathfrak{m}$-adically separated.

Now consider an ascending sequence of finitely generated ideals of $\mathcal{O}_{X,x}$, say

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_i \subset \cdots \subset \mathfrak{m}\mathcal{O}_{X,x}.$$

Using the condition that $\widehat{\mathcal{O}}_{X,x} \cong \widehat{A_{\mathfrak{m}}}$ is noetherian, we see the completion of this sequence terminates, i.e. there exists some $N \gg 0$ such that $\widehat{\mathfrak{a}}_i = \widehat{\mathfrak{a}}_{i+1}$ for all $i > N$.

Finally, by using the injectivity of $\mathcal{O}_{X,x}/\mathfrak{a}_i \to \widehat{\mathcal{O}_{X,x}/\mathfrak{a}_i}$, the original sequence terminates as well, i.e. for all $i > N$, $\mathfrak{a}_{i+1} = \mathfrak{a}_i$. This shows that all ascending sequences in $\mathcal{O}_{X,x}$ are stable, and hence $\mathcal{O}_{X,x}$ is noetherian. □

2. Locally closed immersions of affinoid spaces

2.1. **Locally closed immersions.**

Definition 2.1. A morphism of affinoid K-spaces $\varphi : X' \to X$ is called a *closed immersion* if its corresponding morphism $\varphi^* : \mathcal{O}_X(X) \to \mathcal{O}_{X'}(X')$ of affinoid K-algebras is surjective.

Furthermore, φ is called a *locally closed immersion* (resp. an *open immersion*) if it is injective and for any $x' \in X'$ the induced morphism $\varphi_x^* : \mathcal{O}_{X,\varphi(x')} \to \mathcal{O}_{X',x'}$ is surjective (resp. bijective).

Here are some comments as well as examples to understand the given notion.

- For example, any morphism of affinoid K-spaces $\varphi : X' \to X$ defining X' as an affinoid subdomain of X is an open immersion, due to the transitivity of affinoid subdomains (c.f. Talk 4).
- Due to the previous statements, if φ is a closed immersion then φ is a locally closed immersion.
- Furthermore, any composition of locally closed immersions (resp. closed immersions; open immersions) is an immersion of the same type.

Remark 2.2. Let $\varphi : Y \to X$ be a closed (resp. locally closed; open) immersion. Then for any affinoid subdomain U of X, the induced morphism $\varphi_U : \varphi^{-1}(U) \to U$ is an immersion of the same type.

For this, we only check the case when φ is a closed immersion (since others are simple). Let $Y = \operatorname{Sp} B$ and $X = \operatorname{Sp} A$ with $\varphi^* : A \to B$. Then φ^* is injective. Let $U = \operatorname{Sp} A' \subset X$ be the affinoid subdomain. Then $\varphi^{-1}(U) \cong \operatorname{Sp} B'$ for $B' = A' \hat{\otimes}_A B$. We obtain a Cartesian diagram

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & B. \end{array}$$

Using Proposition 10 of [Bos14, Appendix B], we see $A' \to B'$ is surjective as φ^* is.

Proposition 2.3. *Let $\varphi : X' \to X$ be a locally closed immersion of affinoid K-spaces, where the corresponding homomorphism of affinoid K-algebras is finite. Then φ is a closed immersion.*

Proof. Writing $X' = \operatorname{Sp} A'$ and $X = \operatorname{Sp} A$. The morphism φ induces for every $x \in X'$ a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & A_{\mathfrak{m}_{\varphi(x)}} & \hookrightarrow & \mathcal{O}_{X,\varphi(x)} & \hookrightarrow & \widehat{\mathcal{O}}_{X,\varphi(x)} \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*_{\mathfrak{m}_x}} & & \downarrow{\scriptstyle \varphi^*_x} & & \downarrow{\scriptstyle \widehat{\varphi}^*_x} \\ A' & \longrightarrow & A'_{\mathfrak{m}_x} & \hookrightarrow & \mathcal{O}_{X',x} & \hookrightarrow & \widehat{\mathcal{O}}_{X',x}. \end{array}$$

Since φ is injective, $\mathfrak{m}_x \subset A'$ is the only maximal ideal over $\mathfrak{m}_{\varphi(x)} \subset A$, and therefore we can review $A'_{\mathfrak{m}_x}$ as $A'_{\mathfrak{m}_{\varphi(x)}}$. Since φ^* is finite, we see $\varphi^*_{\mathfrak{m}_x}$ is also finite. Consider $A'_{\mathfrak{m}_x}/\mathfrak{m}_{\varphi(x)}A'_{\mathfrak{m}_x}$: it is finite over $k(\varphi(x)) = A/\mathfrak{m}_{\varphi(x)}$. Then the descending chain $\mathfrak{m}_x^n(A'_{\mathfrak{m}_x}/\mathfrak{m}_{\varphi(x)}A'_{\mathfrak{m}_x})$ will be stable for $n \gg 0$. Again, by Krull's intersection theorem,

$$\bigcap_{n \geqslant 1} \mathfrak{m}_x^n(A'_{\mathfrak{m}_x}/\mathfrak{m}_{\varphi(x)}A'_{\mathfrak{m}_x}) = 0.$$

Then for $n \gg 0$, there must be $\mathfrak{m}_x^n \subset \mathfrak{m}_{\varphi(x)}A'_{\mathfrak{m}_x}$, i.e. the $\mathfrak{m}_x$-adic topology and $\mathfrak{m}_{\varphi(x)}$-adic topology on $A'_{\mathfrak{m}_x}$ coincide. Then we have an isomorphism

$$\widehat{A'_{\mathfrak{m}_x}} \cong \varprojlim_{n \geqslant 0} A'_{\mathfrak{m}_x}/\mathfrak{m}^n_{\varphi(x)}A'_{\mathfrak{m}_x} \cong A'_{\mathfrak{m}_x} \otimes_{A_{\mathfrak{m}_{\varphi(x)}}} \widehat{A_{\mathfrak{m}_{\varphi(x)}}}.$$

Since $\mathcal{O}_{X,\varphi(x)} \to \mathcal{O}_{X',x}$ is surjective, so also is its projective limit $\widehat{\mathcal{O}}_{X,\varphi(x)} \to \widehat{\mathcal{O}}_{X',x}$. From Proposition 1.2,

$$\widehat{A_{\mathfrak{m}_{\varphi(x)}}} \longrightarrow \widehat{A'_{\mathfrak{m}_x}} \cong A'_{\mathfrak{m}_x} \otimes_{A_{\mathfrak{m}_{\varphi(x)}}} \widehat{A_{\mathfrak{m}_{\varphi(x)}}}$$

is surjective as well. Recall that $A_{\mathfrak{m}_{\varphi(x)}} \to \widehat{A_{\mathfrak{m}_{\varphi(x)}}}$ is faithfully flat, so $\widehat{A_{\mathfrak{m}_{\varphi(x)}}} \to A'_{\mathfrak{m}}$ is surjective. By the correspondence of local rings and global rings, it follows that $A \to A'$ is surjective. ∎

Proposition 2.4. *Let $\varphi : X' \to X$ be a morphism of affinoid K-spaces that is an open and closed immersion. Then the image of X' is Zariski open and closed in X; in particular, φ defines a Weierstrass domain in X.*

Proof. Resume on notation as in the preceding proof. Let $\varphi^* : A \to A'$ and $\varphi^*_{\mathfrak{m}_x} : A_{\mathfrak{m}_{\varphi(x)}} \to A'_{\mathfrak{m}_{\varphi(x)}}$ at a point $x \in X'$ as above. Then $\varphi^*_{\mathfrak{m}_x}$ is surjective, since φ^* is surjective. In fact, $\varphi^*_{\mathfrak{m}_x}$ should be bijective. To verify this, we consider

$$\begin{array}{ccc} A_{\mathfrak{m}_{\varphi(x)}} & \hookrightarrow & \mathcal{O}_{X,\varphi(x)} \\ \downarrow & & \downarrow \wr \\ A'_{\mathfrak{m}_x} & \hookrightarrow & \mathcal{O}_{X',x}. \end{array}$$

Then we see $\varphi^*_{\mathfrak{m}_x}$ is injective.

Now, since $\varphi^*_{\mathfrak{m}_x}$ is bijective, there is an element f such that $f(x) \neq 0$ and φ^* induces an isomorphism on $A[f^{-1}] \to A'[f^{-1}]$. Thus, $D(f) \subset \varphi(X')$, and $\varphi(X')$ is Zariski open. On the other hand it is also Zariski closed as φ is a closed immersion. Then there is a decomposition of K-algebras such that $A \simeq A_1 \oplus A_2$ with $\ker \varphi = A_2$ and $A_1 \simeq A'$. So we have a unipotent element $e \in A_2$ with $c \in K$ satisfying $|c| > 1$. It renders that $X(e) = \varphi(X')$ and φ induces the isomorphism $X' \cong X(ce)$. □

2.2. **Runge immersions.** Next we introduce a particular class of locally closed immersions.

Definition 2.5. A morphism of affinoid K-spaces $\varphi : X' \to X$ is called a *Runge immersion* if it is the composition of a closed immersion $X' \to W$ and an open immersion $W \to X$ defining W as a Weierstrass domain in X.

From Remark 2.2 we can immediately deduce:

Remark 2.6. Let $\varphi : X' \to X$ be a Runge immersion of affinoid K-spaces. Then, for any affinoid subdomain U of X the induced morphism $\varphi_U : \varphi^{-1}(U) \to U$ is a Runge immersion, too.

If $\sigma : A \to A'$ is a morphism of affinoid K-algebras, we call finitely many elements $h_1, \dots, h_n \in A'$ a system of affinoid generators of A' over A (with respect to σ) if σ extends to an epimorphism

$$A\langle \zeta_1, \dots, \zeta_n \rangle \longrightarrow A', \quad \zeta_i \longmapsto h_i.$$

Of course, the $h_i \in A'$ are then necessarily power bounded.

Proposition 2.7. *For a morphism of affinoid K-algebras $\sigma : A \to A'$ the following are equivalent:*

(i) *The morphism of affinoid K-spaces $\varphi : \operatorname{Sp} A' \to \operatorname{Sp} A$ associated to σ is a Runge immersion.*
(ii) *$\sigma(A)$ is dense in A'.*
(iii) *$\sigma(A)$ contains a system of affinoid generators of A' over A.*

Proof. If φ is a Runge immersion, $\varphi(A)$ is dense in A', since the corresponding fact is true for closed immersions and for Weierstrass domains. Next, choose a system $h'_1, \dots, h'_n$ of affinoid generators of A' over A. Then, if $\sigma(A)$ is dense in A', we can approximate each h_i' by some $h_i \in \sigma(A)$ in such a way that, using Lemma 8 below, $h_1, \dots, h_n$ will be a system of affinoid generators of A' over A. Finally, assume that $h_1, \dots, h_n \in \sigma(A)$ is a system of affinoid generators of A' over A. Then σ decomposes into the maps

$$A \longrightarrow A\langle h_1, \dots, h_n \rangle \longrightarrow A'$$

where the first one corresponds to the inclusion of $X(h_1, \dots, h_n)$ as a Weierstrass domain in $X = \operatorname{Sp} A$ and where the second is surjective and, hence, corresponds to a closed immersion $\operatorname{Sp} A' \to X(h_1, \dots, h_n)$. Thus, φ is a Runge immersion. □

As a consequence we see that the composition of finitely many Runge immersions or, more specifically, closed immersions and inclusions of Weierstrass domains, yields a Runge immersion again.

Lemma 2.8. *Consider a morphism of affinoid K-algebras $\sigma : A \longrightarrow A'$ and a system $h' = (h_1', \dots, h_r')$ of affinoid generators of A' over A. Fix a residue norm on A and consider on A' the residue norm via the epimorphism*

$$\pi' : A\langle \zeta \rangle \longrightarrow A', \quad \zeta \longmapsto h',$$

where we endow $A\langle \zeta \rangle = A\langle \zeta_1, \dots, \zeta_n \rangle$ with the Gauß norm derived from the given residue norm on A. Then any system $h = (h_1, \dots, h_n)$ in A' such that $|h_i' - h_i| < 1$ for all i, yields a system of affinoid generators of A' over A.

Proof. Since $|h_i'| \leqslant 1$, due to our assumption, we have $|h_i| \leqslant 1$ for all i and therefore can consider the morphism

$$\pi : A\langle \zeta \rangle \longrightarrow A', \quad \zeta \longmapsto h.$$

Let $\varepsilon = \max_{i=1,\dots,n} |h_i' - h_i|$ so that $\varepsilon < 1$. It is enough to show for any element $g \in A' = \operatorname{im} \pi'$ that there is some $f \in A\langle \zeta_1, \dots, \zeta_n \rangle$ satisfying $|f| = |g|$ and $|\pi(f) - g| \leqslant \varepsilon |g|$. Then an iterative approximation argument shows that π is surjective.

Thus, start with an element $g \in A'$ and choose a π'-inverse $f = \sum_{\nu \in \mathbb{N}^n} a_\nu \zeta^\nu$ in $A\langle \zeta \rangle$ with coefficients $a_v \in A$; we may assume $|f| = |g|$ by [Bos14, Proposition 3.1.5] (c.f. Talk 3). Then

$$\begin{aligned} |\pi(f) - g| &= \left| \sum_{\nu \in \mathbb{N}^n} a_v h^\nu - \sum_{\nu \in \mathbb{N}^n} a_v h'^\nu \right| \\ &= \left| \sum_{\nu \in \mathbb{N}^n} a_\nu \left(h^\nu - h'^\nu \right) \right| \leqslant \varepsilon \max_{v \in \mathbb{N}^n} |a_\nu| = \varepsilon |g|, \end{aligned}$$

as required. □

2.3. **The extension lemma.** Next, we want to derive a certain extension lemma for Runge immersions. To do this, let K_a be an algebraic closure of K and write K_a^* for its multiplicative group, as well as $|K_a^*|$ for the corresponding value group. Then $|K_a^*|$ consists of all real numbers $\alpha > 0$ such that there is some integer $s > 0$ satisfying $\alpha^s \in K^*$. Furthermore, let $X = \operatorname{Sp} A$ be an affinoid K-space and consider functions $f_1, \dots, f_r, g \in A$ generating the unit ideal. Then, for any $\varepsilon \in |K_a^*|$, we may consider the subset

$$X_\varepsilon = \{x \in X; |f_j(x)| \leqslant \varepsilon |g(x)|, j = 1, \dots, r\} \subset X.$$

If $\varepsilon^s = |c|$ for some $c \in K^*$, the set X_ε is characterized by the estimates

$$\left|f_j^s(x)\right| \leqslant |c^s(x)|, \quad j = 1, \ldots, r,$$

and therefore defines a rational subdomain in X. Given a morphism of affinoid K-spaces $\varphi : X' \longrightarrow X$, we set $X'_\varepsilon = \varphi^{-1}(X_\varepsilon)$ and consider the morphism $\varphi_\varepsilon : X'_\varepsilon \longrightarrow X_\varepsilon$ induced by φ.

Theorem 2.9 (Extension lemma)**.** *Assume that the morphism $\varphi_{\varepsilon_0} : X'_{\varepsilon_0} \longrightarrow X_{\varepsilon_0}$ defined as before is a Runge immersion for some $\varepsilon_0 \in |K_a^*|$. Then there is an $\varepsilon \in |K_a^*|, \varepsilon > \varepsilon_0$, such that $\varphi_\varepsilon : X'_\varepsilon \longrightarrow X_\varepsilon$ is a Runge immersion as well.*

Proof. Write $X = \operatorname{Sp} A$ and $X' = \operatorname{Sp} A'$, as well as $X_\varepsilon = \operatorname{Sp} A_\varepsilon$ and $X'_\varepsilon = \operatorname{Sp} A'_\varepsilon$ for $\varepsilon \in |K_a^*|$. Replacing X by $X_{\varepsilon'}$ and X' by $X'_{\varepsilon'}$ for some $\varepsilon' \in |K_a^*|, \varepsilon' > \varepsilon_0$, we may assume that all X_ε and X'_ε are Weierstraß domains in X and X', respectively. Then, for $\varepsilon \in |K_a^*|, \varepsilon \geqslant \varepsilon_0$, we have a canonical commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varphi^*} & A' \\
\downarrow & & \downarrow \\
A_\varepsilon & \xrightarrow{\varphi_\varepsilon^*} & A'_\varepsilon \\
\downarrow & & \downarrow \\
A_{\varepsilon_0} & \xrightarrow{\varphi_{\varepsilon_0}^*} & A'_{\varepsilon_0}
\end{array}$$

where the vertical maps all have dense images, since, on the level of affinoid spaces, they correspond to inclusions of Weierstraß domains. Now let $h' = (h'_1, \ldots, h'_n)$ be a system of affinoid generators of A' over A. Then h' gives rise to a system h'_ε of affinoid generators of A'_ε over A_ε, as well as to a system h'_{ε_0} of affinoid generators of A'_{ε_0} over A_{ε_0}.

Let us restrict ourselves for a moment to values $\varepsilon \in |K^*|$. In particular, we assume $\varepsilon_0 \in |K^*|$. Fixing a residue norm on A, we consider on A' the residue norm with respect to the epimorphism

$$\pi : A\left\langle \zeta_1, \ldots, \zeta_n \right\rangle \longrightarrow A', \quad \zeta_i \longmapsto h'_i,$$

and on each A_ε the residue norm with respect to the epimorphism

$$p_\varepsilon : A\left\langle \varepsilon^{-1}\eta_1, \ldots, \varepsilon^{-1}\eta_r \right\rangle \longrightarrow A_\varepsilon, \quad \eta_j \longmapsto \frac{f_j}{g},$$

where, strictly speaking, the element ε in the expression $\varepsilon^{-1}\eta_j$ has to be replaced by a constant $c \in K$ with $|c| = \varepsilon$ and where the elements $\varepsilon^{-1}\eta_j$ have to be viewed as variables. Then we can introduce on any A'_ε the residue norm via the epimorphism

$$\pi_\varepsilon : A_\varepsilon\left\langle \zeta_1, \ldots, \zeta_n \right\rangle \longrightarrow A'_\varepsilon, \quad \zeta_i \longmapsto h'_i.$$

The latter equals the residue norm that is derived from the one of A via the epimorphism

$$\tau_\varepsilon : A\left\langle \zeta_1, \ldots, \zeta_n, \varepsilon^{-1}\eta_1, \ldots, \varepsilon^{-1}\eta_r \right\rangle \longrightarrow A'_\varepsilon, \quad \zeta_i \longmapsto h'_i, \quad \eta_j \longmapsto \frac{f_j}{g}$$

satisfying

$$\ker \tau_\varepsilon = (\ker \pi, g\eta_1 - f_1, \ldots, g\eta_r - f_r).$$

Now choose a system $h = (h_1, \dots, h_n)$ of elements in A', having φ^*-inverses in A, and whose images in A'_{ε_0} satisfy

$$(*) \qquad \left| h'_i|_{X'_{\varepsilon_0}} - h_i|_{X'_{\varepsilon_0}} \right| < 1, \quad i = 1, \dots, n.$$

The latter is possible, since the image of A is dense in A'_{ε_0}, due to the fact that X_{ε_0} is a Weierstraß domain in X and $\varphi_{\varepsilon_0} : X'_{\varepsilon_0} \longrightarrow X_{\varepsilon_0}$ is a Runge immersion. Then it follows from Lemma 2.8 that h gives rise to a system of affinoid generators of A'_{ε_0} over A_{ε_0}.

In order to settle the assertion of the Extension Lemma, it is enough to show that, in fact,

$$|h'_i|_{X'_\varepsilon} - h_i|_{X'_\varepsilon} | < 1, \quad i = 1, \dots, n,$$

for some $\varepsilon > \varepsilon_0$. Then, using Lemma 2.8 again, $h|_{X'_\varepsilon}$ is a system of affinoid generators of A'_ε over A_ε belonging to the image of A_ε, and it follows from Proposition 2.7 that $\varphi_\varepsilon : X'_\varepsilon \longrightarrow X_\varepsilon$ is a Runge immersion in this case.

To abbreviate, let $d_\varepsilon = h'_i|_{X'_\varepsilon} - h_i|_{X'_\varepsilon} \in A'_\varepsilon$ for any $i \in \{1, \dots, n\}$. Furthermore, fix $\varepsilon_1 \in |K^*|$ with $\varepsilon_1 > \varepsilon_0$ and choose an element $g_{\varepsilon_1} \in A\left\langle \zeta, \varepsilon_1^{-1}\eta \right\rangle$ with $\tau_{\varepsilon_1}(g_{\varepsilon_1}) = d_{\varepsilon_1}$ where $\zeta = (\zeta_1, \dots, \zeta_n)$ and $\eta = (\eta_1, \dots, \eta_r)$. For $\varepsilon \leqslant \varepsilon_1$, let g_ε be the image of g_{ε_1} in $A\left\langle \zeta, \varepsilon^{-1}\eta \right\rangle$ so that $\tau_\varepsilon(g_\varepsilon) = d_\varepsilon$ for all $\varepsilon \leqslant \varepsilon_1$. Now, by the choice of h_i, we have $|d_{\varepsilon_0}| < 1$. Thus, using [Bos14, Proposition 3.1.5] (c.f. Talk 3), there is an element

$$g_0 \in \ker \tau_{\varepsilon_0} = (\ker \pi, g\eta_1 - f_1, \dots, g\eta_r - f_r)\, A\left\langle \zeta, \varepsilon_0^{-1}\eta \right\rangle$$

such that $|g_{\varepsilon_0} + g_0| < 1$. Approximating functions in $A\left\langle \zeta, \varepsilon_0^{-1}\eta \right\rangle$ by polynomials in $A\langle\zeta\rangle\left[\varepsilon_0^{-1}\eta\right]$, we may assume that g_0 is induced by an element

$$g_1 \in \ker \tau_{\varepsilon_1} = (\ker \pi, g\eta_1 - f_1, \dots, g\eta_r - f_r)\, A\left\langle \zeta, \varepsilon_1^{-1}\eta \right\rangle$$

But then we may replace from the beginning g_{ε_1} by $g_{\varepsilon_1} - g_1$ and thereby assume $|g_{\varepsilon_0}| < 1$. Now let

$$g_{\varepsilon_1} = \sum_{\mu \in \mathbb{N}^n, \nu \in \mathbb{N}^r} a_{\mu\nu} \zeta^\mu \eta^\nu \in A\left\langle \zeta, \varepsilon_1^{-1}\eta \right\rangle$$

with coefficients $a_{\mu\nu} \in A$. Since $|g_{\varepsilon_0} < 1|$, we get $\max_{\mu \in \mathbb{N}^n, v \in \mathbb{N}^r} |a_{\mu\nu}|\, \varepsilon_0^{|\nu|} < 1$. Passing from ε_0 to a slightly bigger ε (not necessarily contained in $|K^*|$), we still have

$$|g_\varepsilon| = \max_{\mu \in \mathbb{N}^n, \nu \in \mathbb{N}^r} |a_{\mu\nu}|\, \varepsilon^{|\nu|} < 1$$

for $\varepsilon > \varepsilon_0$ sufficiently close to ε_0. Thus, if such ε exist in $|K^*|$, the series g_ε is a well-defined element in $A\left\langle \zeta, \varepsilon^{-1}\eta \right\rangle$ satisfying $|d_\varepsilon| \leqslant |g_\varepsilon| < 1$ as required in $(*)$. This settles the assertion of the Extension Lemma in the case where $\varepsilon_0 \in |K^*|$ and the valuation on K is non-discrete.

In the general case, we can always enlarge the value group $|K^*|$ by passing to a suitable finite algebraic extension L/K. This way, we can assume $\varepsilon_0 \in |L^*|$ and, in addition, that the last step in the above argumentation works for some $\varepsilon > \varepsilon_0$ contained in $|L^*|$. In other words, the assertion of the Extension Lemma holds after replacing the base field K by a suitable finite algebraic extension L in the sense that we apply to our situation the base change functor

$$\operatorname{Sp} A \longmapsto \operatorname{Sp} A \hat{\otimes}_K L$$

where $\hat{\otimes}$ is the completed tensor product of [Bos14, Appendix B]. Thus, it is enough to show that a morphism of affinoid K-spaces $X' \longrightarrow X$ is a Runge immersion if the corresponding morphism of affinoid L-spaces $X'\hat{\otimes}_K L \longrightarrow X\hat{\otimes}_K L$ has this property or, equivalently, that a morphism of affinoid K-algebras $A \longrightarrow A'$ has dense image if the corresponding morphism of affinoid L-algebras $A\hat{\otimes}_K L \longrightarrow A'\hat{\otimes}_K L$ has dense image. However, the latter is easy to see. Since the completed tensor product commutes with finite direct sums, see the discussion following Proposition 2 of [Bos14, Appendix B], it follows that the canonical morphism $A \otimes_K L \longrightarrow A\hat{\otimes}_K L$ is bijective for any affinoid K-algebra A and any finite extension L/K. Now consider a morphism of affinoid K-algebras $\sigma : A \longrightarrow A'$, and let $A'' \subset A'$ be the closure of $\sigma(A)$. Then the morphism $\sigma \otimes_K L : A \otimes_K L \longrightarrow A' \otimes_K L$ factors through the closed subalgebra $A'' \otimes_K L \subset A' \otimes_K L$. If $\sigma \otimes_K L$ has dense image, we see that $A'' \otimes_K L$ coincides with $A' \otimes_K L$ and, hence, by descent, that the same is true for A'' and A'. Thus, we are done. □

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