Grothendieck Topology and the Notion of Rigid Spaces.

(A). Generally, a grothendieck topology T consists of:

(i). a category $\underline{\mathcal{C}}$, and a ~~family~~ set $Cov(T)$ of families of morphisms $(U_i \to U)_{i\in I}$ in $\underline{\mathcal{C}}$, s.t.

(i) If $U \to V$ is an isomorphism, then $(U \to V) \in Cov(T)$

(ii) If $(U_i \to U)_{i\in I} \in Cov(T)$, $(V_{ij} \to U_i)_{j\in J} \in Cov(T)$, then $(V_{ij} \to U_i) \in Cov(T)$

(iii) If $(U_i \to U) \in Cov(T)$, $V \to U$ a morphism in $\underline{\mathcal{C}}$, then

$(U_i \times_U V \to V) \in Cov(T)$

In our case, $\underline{\mathcal{C}}$ is ~~often~~ a category consists of certain subsets of a set X, with inclusions as morphisms, intersection as fiber products.

<u>Def</u>. X a given set, we call X is a G-topological space when we are given:

(i) A system S of ~~open~~ subsets of X, called admissible open subsets.

(ii) A family $\{Cov\, U\}_{U\in S}$ of systems of coverings, called admissible coverings, where $Cov\, U$ for $U \in S$ contains coverings $\{U_i \to U\}$ by sets in S.

s.t. Admissible open subsets and admissible coverings satisfy the condition of Grothendieck topology. (with $U_i \times_U V = U_i \cap V$)

<u>Def</u>. Weak Grothendieck topology.

For an affinoid space X, admissible open subsets are affinoid subdomains, admissible coverings are $(U_i \to U)_{i\in I}$, s.t. $U = \bigcup_{i\in I} U_i$ and I is finite.

Then for morphisms of affinoid $Z \xrightarrow{\varphi} X$. φ is continuous w.r.t. this toplogy. here continuous "means" φ^{-1} takes admissible open/covering to admissible open/covering.

~~Def. Strong Grothendieck topology~~

<u>Rmk</u>. Tate's acyclicity thm implies that $\mathcal{O}_X$ is a sheaf w.r.t. weak Grothendieck topology.

Def. Strong Grothendieck Topology. X an affinoid sp. $U = \bigcup_{i\in I} U_i$

1. A subset $U \subseteq X$ is admissible open if $\exists$ covering ~~$(U_i \to U)$~~ by affinoid ~~spaces~~ subdomains $U_i \subseteq X$, s.t. for all morphisms of affinoid K-spaces $Z \xrightarrow{\varphi} X$, ~~set~~ with $\varphi(Z) \subseteq U$, ~~$(\varphi^{-1}(U_i))_{i\in I}$~~ the covering $Z = \bigcup_{i\in I} \varphi^{-1}(U_i)$ admits a finite refinement by affinoid subdomains

2. A covering $V = \bigcup_{j\in J} V_j$ is called admissible if $\varphi: Z \to X$ with $\varphi(Z) \subseteq V$, the covering $Z = \bigcup_{j\in J} \varphi^{-1}(V_j)$ admits a finite refinement by affinoid subdomains.

(B) Rmk (BGR). (Slightly finer)

Def. T and T' are G-topologies on a fixed ~~space~~ set, X. T' is called slightly finer than T, if (i) T' is finer than T

(ii) The T-open subsets of X form a basis of T' (i.e. if U is T'-open, $\exists (U_i \to U)$ is a T'-covering, and $U_i \in T$)

(iii) For each T'-covering $(U_i \to U)$ ~~with~~ of a T-open subset $U \subseteq X$, $\exists$ a T-covering refines $(U_i \to U)$ (Prop 5.2/4)

On an affinoid space X, strong grothendieck ~~topology~~ is slightly finer than weak grothendieck topology.

Conditions on G-topology:

(G_0) ϕ and X are admissible open

(G_1) $U \subseteq X$ admissible open, $V \subseteq U$ a subset. Assume that $\exists$ admissible covering $(U_i \to U)$, s.t. $V \cap U_i$ is admissible open in X. then V is an admissible open

(G_2) $(U_i \to U)$ is ~~an admissible~~ a covering of an admissible open set $U \subseteq X$, with U_i admissible open in X, Assume $(U_i \to U)$ admits a refinement that is admissible, Then $(U_i \to U)$ is admissible.

__Facts__. Let X be a set and T a G-topology on X, then there exists a unique finest G-topology T' on X among all the G-topologies slightly finer than T; and T' satisfies (G_1), (G_2); when T satisfies (G_0) T' also satisfies (G_0) [BGR] P.339-340.

We leave the details to Appendix.

The construction of the "unique finest topology among those slightly finer than $\mathcal{T}$". is ~~the~~ "universal": If $\varphi: X \to Y$ is a morphism of affinoid space,

then φ is continuous w.r.t. the weak topology

$\Rightarrow$ continuous w.r.t. the strong topology.

(Also can be checked directly)

Propostion. Let X be an affinoid K-space, for $f \in \mathcal{O}_X(X)$, consider

$$U = \{x : |f(x)| < 1\}$$

$$U' = \{x : |f(x)| > 1\}$$

$$U'' = \{x : |f(x)| > 0\}$$

Then Any finite union of this type is admissible open. Any finite covering by finite union of this type is admissible

Cor Since ~~type U''~~ any Zariski open subset is a finite union of subsets of type U'',

~~Zari~~ Strong topology is finer than Zariski topology.

proof. Here we prove the case ~~for U~~ of U.

~~Consider~~ Choose $\varepsilon_v \in \sqrt{|K^*|}$, s.t. $\varepsilon_v < 1$ and $\lim_{v\to\infty} \varepsilon_v = 1$. $\sqrt{|K^*|} = \{s \in \mathbb{R}_{>0} : s^n \in |K^*|$ for some $n\}$

we have $U = \bigcup_{v=0}^{\infty} X(\varepsilon_v^{-1} f)$ ~~and we prove this is gives the coveri~~

$$X(g) = \{x \in X : |g(x)| \le 1\}$$

$X(\varepsilon_v^{-1} f) = X(c^{-1} f^r)$, here $|\underbrace{\varepsilon_v^r}_{c}| \in |K^*|$

For $\varphi: Z \to X$, with $\varphi(Z) \subseteq U$,

we have $|(\varphi^*(f))(z)| = |f(\varphi(z))| < 1 \xrightarrow[\text{principle}]{\text{maximum}} |\varphi^*(f)|_{sup} < 1$

while $Z = \bigcup_{v=0}^{\infty} Z(\varepsilon_v^{-1} \varphi^*(f))$

so ~~most~~ of only finitely many v s.t. $Z(\varepsilon_v^{-1}\varphi^*(f)) \ne Z$

certainly admits a finite subcover. This shows that U is admissible open

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Now, why do we define the strong topology?

Every sheaf on weak topology can be extended (functorially) to a sheaf on strong topology.

On the other hand, strong topology behave well with taking "subspace topology"!

Prop. Let T be a G-topology on X satisfying condition $(G_0)-(G_2)$
~~then~~ Let $(X_i \to X)$ be an admissible covering.
Then (i) $U \subseteq X$ a subset is admissible open iff $U \cap X_i$ is admissible open
(ii) $(U_i \to U)$ a covering of some admissible open set $U \subseteq X$
is admissible iff $(U_i \cap X_j \to U \cap X_j)$ is admissible.

Prop. Let X be a set, $(X_i)_{i\in I}$ a covering of X. T_i are G-topologies on X_i, s.t. T_i satisfies $(G_0)-(G_2)$ and $X_i \cap X_j$ is T_i-open; and T_i and T_j restricts to the same Grothendieck topology on $X_i \cap X_j$. Then there exists a unique G-topology T on X, s.t.

(i) X_i is T-open in X, and T induces T_i on X_i
(ii) T satisfies $(G_0)-(G_2)$
(iii) $(X_i \to X)_{i\in I}$ is a T-covering.

Proof. Define T as following:

① A subset U is T-open if each $X_i \cap U$ is T_i-open
② A covering $(U_i \to U)$ of T-open subsets $U_i \subseteq X$ is a T-covering if
$(X_i \cap U_j \to U \cap X_i)$ is T_i-open □

From the above propositions, one can sees the need of certain completeness property $(G_0)-(G_2)$.

(c) Sheaves.

A presheaf on a G-topological space X is a contravariant functor from Category of admissible opens to groups/rings/etc.

A sheaf is a presheaf $\mathcal{F}$ s.t.

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$$ is exact for admissible coverings.

Def. (Stalk). For a G-top. space X, ~~$x \in X$~~, $\mathcal{F}$ a presheaf on X. $x \in X$ a point, define $\mathcal{F}_x = \varinjlim_{\substack{x \in U \\ U \text{ admissible}}} \mathcal{F}(U)$ to be the stalk of $\mathcal{F}$ at x.

~~Rmk. If T is slightly finer than~~

Def. (Sheafification) $\mathcal{F}$ a ~~presheaf~~ presheaf on a G-topological sp. X.
$\exists$ A sheaf $\mathcal{F}'$ and a morphism $\mathcal{F} \to \mathcal{F}'$, s.t.

$$\operatorname{Hom}_{P}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{S}(\mathcal{F}', \mathcal{G}) \text{ for any sheaf } \mathcal{G}.$$

and this $\mathcal{F}'$ is unique up to isomorphism. (Also thm).

Sketch of proof. Define $\check{H}^0(U, \mathcal{F}) = \varinjlim_{\mathfrak{U}} H^0(\mathfrak{U}, \mathcal{F})$

$\mathcal{F}^+ = (U \mapsto \check{H}^0(U, \mathcal{F}))$ and $\mathcal{F} \to \mathcal{F}^+$ given by the canonical map $\mathcal{F}(U) \to H^0(\mathfrak{U}, \mathcal{F})$

Then ① $\mathcal{F}^+$ is separated

② If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf. $\square$

Prop. T' is slightly finer than T, then a sheaf $\mathcal{F}$ on T can be (functorially) extended to a T'-sheaf $\mathcal{F}'$, and this extension is unique up to isomorphism.

Sketch $U \mapsto \varinjlim_{\mathfrak{U}} H^0(\mathfrak{U}, \mathcal{F})$, where $\mathfrak{U}$ runs over admissible T'-covering such that $\mathfrak{U} = (U_i)$ U_i are T-open. It is easy to see $\mathcal{F}'$ extends $\mathcal{F}$. $\square$

Cor. There is an extension of $\mathcal{O}_X$ to the strong grothendieck topology on an affinoid space X. $\square$

(D) Rigid Spaces.

Def. A G-ringed k-space is a pair $(X, \mathcal{O}_X)$ consisting of a G-topological space X and a sheaf of k-algebras. $(X, \mathcal{O}_X)$ is called a locally G-ringed k-space if all the stalks $\mathcal{O}_{X,x}$ are local rings.

For an affinoid space, since the ~~local ring~~ stalk (of weak topology) is local and strong topology is slightly finer than the weak topology, ~~$\mathcal{O}_X$ is~~ $(X, \mathcal{O}_X)$ is a locally G-ringed k-space.

<u>Def</u>. A morphism of G-ringed k-space $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $\varphi : X \to Y$ and a compatible system of maps $\mathcal{O}_Y(V) \to \mathcal{O}_X(\varphi^{-1}(V))$.

Further, when X, Y are locally G-ringed, ~~we requi~~ a ~~morf~~ morphism of locally G-ringed ~~k~~-spaces require $\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ to be a local homomorphism.

<u>Prop</u>. Let X and Y be affinoid spaces. Then

$$\mathrm{Hom}_{\text{Locally ringed space}}\big((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\big) = \mathrm{Hom}_{\text{Algebra}}\big(\mathcal{O}_Y(Y), \mathcal{O}_X(X)\big)$$

~~proof.~~ ~~Step 1. For $\varphi_Y^* : \mathcal{O}_Y(X) \to \mathcal{O}_Y(Y)$, gives a morphism of affinoid space. For an affinoid~~

<u>Step 1</u>. Fix a morphism of affinoid k-spaces $\varphi : X \to Y$, ~~φ gives~~

① for an affinoid $V \subseteq Y$, $\varphi^{-1}(V) \subseteq X$ is an affinoid and φ induces a morphism of affinoid k-spaces $\varphi_V : \varphi^{-1}(V) \to V$, i.e. $\mathcal{O}_Y(V) \xrightarrow{\varphi_V^*} \mathcal{O}_X(\varphi^{-1}(V))$

② for an admissible open $V \subseteq Y$, choose an admissible affinoid covering $(V_i \to V)$ of V. then we can obtain φ_V^* from

$$\begin{array}{ccccc}
\mathcal{O}_Y(V) & \longrightarrow & \prod \mathcal{O}_Y(V_i) & \rightrightarrows & \prod \mathcal{O}_Y(V_i \cap V_j) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_X(\varphi^{-1}(V)) & \longrightarrow & \cdots & \rightrightarrows & \cdots
\end{array}$$

Then (φ, φ^*) is a morphism of G-ringed k-spaces

Step 2. $m_{\varphi(x)} = (\varphi_Y^*)^{-1}(m_x)$ for $x \in X$, and maximal ideal of $\mathcal{O}_{Y,\varphi(x)}$ $(\mathcal{O}_{X,x})$ is generated by $m_{\varphi(x)}$ (m_x), so $\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is a local hom.

<u>Step 3.</u> $(\varphi,\varphi^*): (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y) \underset{1:1}{\longleftrightarrow} \varphi_Y^*: \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$

Only need to prove: for a $\sigma^*: \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$, there is a unique $(\varphi,\varphi^*): (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$, s.t. $\varphi_Y^* = \sigma^*$

① φ is unique:

$$\begin{tikzcd} \mathcal{O}_Y(Y) \arrow[r, "\sigma^*=\varphi_Y^*"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\ \mathcal{O}_{Y,\varphi(x)} \arrow[r, "\varphi_x^*"] & \mathcal{O}_{X,x} \end{tikzcd}$$

and $\mathcal{O}_X(X)/\underline{m}_x \xrightarrow{\sim} \mathcal{O}_{X,x}/\underline{m}_x\mathcal{O}_{X,x}$

implies $\underline{m}_{\varphi(x)} = (\sigma^*)^{-1}\underline{m}_x \Rightarrow \varphi(x)$ is determined by σ^*

② φ_V^* is unique: only need to check the case V is an affinoid subdomain

$$\begin{tikzcd} \mathcal{O}_Y(Y) \arrow[r, "\varphi_Y^*=\sigma^*"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\ \mathcal{O}_Y(V) \arrow[r, "\varphi_V^*"] & \mathcal{O}_X(\varphi^{-1}(V)) \end{tikzcd}$$

φ_V^* is unique by the universal property of $V \hookrightarrow Y$.

$\square$

<u>Rmk.</u> An inclusion of an affinoid subdomain $U \hookrightarrow X$ give rise to an open immersion $(U,\mathcal{O}_U) \to (X,\mathcal{O}_X)$

This allow us to globalize affinoid.

<u>Def.</u> A rigid K-space is a locally G-ringed K-space $(X,\mathcal{O}_X)$.

s.t. (i) G-topology of X satisfy $(G_0)-(G_2)$

(ii) X admits an admissible cover $(X_i)_{i\in I}$, s.t. $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid subdomain.

Pasting and Glueing

<u>Prop.</u> X_i, open subspace $X_{ij} \subseteq X_i$, and isomorphisms $X_{ij} \xrightarrow{\varphi_{ij}} X_{ji}$,

s.t. $\varphi_{ii} = id$, $\varphi_{ij} = \varphi_{ji}^{-1}$, $\varphi_{ijk}: X_{ij} \cap X_{ik} \xrightarrow{\sim} X_{ji} \cap X_{jk}$ satisfy cocycle condition

Then X_i glue up to a rigid space X.