Rigid Analytification

Recall X be a proj variety$/\mathbb{C}$, X^{an} its analytification.
$\overset{\shortparallel}{X(\mathbb{C})}$

$\mathcal{F}, \mathcal{G}$ be a coherent sheaf on X. Then

$\mathcal{F}^{an} = f^*\mathcal{F}$
$f: X^{an} \to X$

(a) $H^i(X, \mathcal{F}) \cong H^i(X^{an}, \mathcal{F}^{an})$;

(b) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}^{an}, \mathcal{G}^{an})$;

(c) Any coherent sheaf on X^{an} is the analytification of a coherent sheaf on X.

Goal X locally of finite type over K, construct $X^{rig} \to X$, which gives GAGA functor: $\mathrm{Sch}_K \to \mathrm{Ansp}_K$, $X \mapsto X^{rig}$.

Example $\underline{\mathbb{P}_K^{n,rig} = \bigcup_i \mathrm{Sp}\, K\langle \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \rangle}$

$n=1$
$y = x_0/x_1$, $|y| \leq 1$
$|y| \geq 1$

$c \in K, |c| > 1$ $\quad \mathbb{A}_K^{n,rig} = \bigcup_i \mathrm{Sp}\, K\langle c^{-i}x_1, \ldots, c^{-i}x_n \rangle$, gluing maps are the inclusions.

$$K\langle c^{-i}x_1, \ldots, c^{-i}x_n \rangle = T_{n,|c|^i} =: T_n^{(i)}$$

Def $(Z, \mathcal{O}_Z)$ scheme of locally finite type over K. A rigid analytification of $(Z, \mathcal{O}_Z)$ is a rigid K-space $(Z^{rig}, \mathcal{O}_{Z^{rig}})$, with a morphism of locally G-ringed K-spaces $\iota: (Z^{rig}, \mathcal{O}_{Z^{rig}}) \to (Z, \mathcal{O}_Z)$ satisfying: given rigid K-space $(Y, \mathcal{O}_Y)$ and morphism of locally G-ringed K-spaces $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, the latter factors through ι via a unique morphism of rigid K-spaces $(Y, \mathcal{O}_Y) \to (Z^{rig}, \mathcal{O}_{Z^{rig}})$

i.e. The functor $\mathrm{Ansp}_K \to \mathrm{Sets}$, $(Y, \mathcal{O}_Y) \to \mathrm{Hom}_{G\text{-}\mathrm{LocRigSp}}((Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z))$ is representable in Ansp_K.

Prop (i) Rigid analytification exists for schemes locally of finite type over K. (ii) Moreover, the underlying map of sets of $\iota: Z^{rig} \to Z$ identifies points of Z^{rig} with closed points Z.

Pf. WMA Z is affine by glueing. Write $Z = \operatorname{Spec} K[x_1, \dots, x_n]/I$.

(i) First construct Z^{rig}. Note

$$T_n^{(0)}/(I) \leftarrow T_n^{(1)}/(I) \leftarrow T_n^{(2)}/(I) \leftarrow \cdots \leftarrow K[\underline{x}]/I$$

$$\rightsquigarrow \operatorname{Max} T_n^{(0)}/(I) \hookrightarrow \operatorname{Max} T_n^{(1)}/(I) \hookrightarrow \operatorname{Max} T_n^{(2)}/(I) \hookrightarrow \cdots \hookrightarrow \operatorname{Max} K[\underline{x}]/I$$

$$K\langle c^{-i}x_1, \dots, c^{-i}x_n \rangle$$
$$\|$$
$$T_{n,|c|^i} =: T_n^{(i)}$$

Set $Z^{rig} = \bigcup_i \operatorname{Sp} T_n^{(i)}/(I)$.

[Lem. Y rigid K-space, Z affine K-scheme, then

$$\{(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)\} \leftrightarrow \{\mathcal{O}_Z(Z) \to \mathcal{O}_Y(Y)\}$$]

$$K[\underline{x}]/I \to T_n^{(i)}/(I) \rightsquigarrow \mathcal{O}_Z(Z) \to \mathcal{O}_{Z^{rig}}(Z^{rig})$$

$$\overset{\text{Lem}}{\rightsquigarrow} \iota: (Z^{rig}, \mathcal{O}_{Z^{rig}}) \to (Z, \mathcal{O}_Z).$$

Now for a $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, WMA Y affinoid, then the map correspond to a K-morphism $\sigma: K[\underline{x}]/I \to B = \mathcal{O}_Y(Y)$.

It then suffices to show, for $i \gg 0$, σ factor through $T_n^{(i)}/(I)$ for a unique $T_n^{(i)}/(I) \to B$. For this,

Choose i s.t. $\bar{x}_i \in K[\underline{x}]/I$ satisfy $|\sigma(\bar{x}_i)|_{sup} \le |c|^i$ in B.

Then $\tilde{\sigma}: K[\underline{x}] \to B$ extends uniquely to $T_n^{(i)}$, and σ factor through

$T_n^{(i)}/(I)$ uniquely. $\square$

(ii) Remain to show $\mathrm{Max}\, K[\underline{x}]/(I) = \bigcup_i \mathrm{Max}\, T_n^{(i)}/(I)$. WMA $I=0$.
Then the result follows from:

<u>(a)</u> Let $m \subset K\langle \underline{x}\rangle$ max'l ideal. Then $m' = m \cap K[\underline{x}]$ is a max'l ideal in $K[\underline{x}]$ satisfying $m = m' K\langle \underline{x}\rangle$.

<u>(b)</u> Given a max'l ideal $m' \subset K[\underline{x}]$, there is an index i_0 s.t. $m' K\langle c^{-i}\underline{x}\rangle$ is max'l in $T_n^{(i)}$ for all $i \geq i_0$.

<u>Prf of (a)</u>

$$\begin{tikzcd} K[\underline{x}] \arrow[r, hook] \arrow[d] & K\langle \underline{x}\rangle \arrow[d] \\ K[\underline{x}]/m' \arrow[r, hook] & \underbrace{K\langle \underline{x}\rangle/m}_{\text{finite ext'n of } K} \end{tikzcd} \Rightarrow m' \text{ max'l}$$

For $m = m' K\langle \underline{x}\rangle$, note in

$$\begin{tikzcd} K[\underline{x}]/m' \arrow[r, two heads] \arrow[d, equal] & K\langle \underline{x}\rangle/m' K\langle \underline{x}\rangle \arrow[d] \\ K[\underline{x}]/m' \arrow[r, two heads] & K\langle \underline{x}\rangle/m \end{tikzcd}$$

as $K[\underline{x}]$ is dense $K\langle \underline{x}\rangle$, and fin. dim. vecsps are complete, horizontal maps are surjective. Combined with above, we see $K[\underline{x}]/m' \to K\langle \underline{x}\rangle/m$ is bijective. Hence so is $K[\underline{x}]/m' \to K\langle \underline{x}\rangle/m' K\langle \underline{x}\rangle$, therefore $K\langle \underline{x}\rangle/m' K\langle \underline{x}\rangle \to K\langle \underline{x}\rangle/m$.

<u>Prf of (b)</u>. Consider a max'l $m' \subset K[\underline{x}]$, then $K[\underline{x}]/m'$ is a finite extension of K. Choose i_0 s.t. all $\bar{z}_j \in K[\underline{x}]/m'$ have absolute value $|\bar{z}_j| \leq |c|^{i_0}$, thus $K[\underline{x}] \to K[\underline{x}]/m'$ factors uniquely through $T_n^{(i)} = K\langle c^{-i}\underline{x}\rangle$ for $i \geq i_0$. $\square$

Cor Rigid analytification defines a funtor from the category of K-schemes locally of finite type to the category of rigid K-space. This is the so called GAGA-functor.

This functor is faithful but not fullyfaithful.

$$f_i: Y \to Z \rightsquigarrow f_i^{an}: Y^{an} \to Z^{an}, \quad f_1^{an} = f_2^{an}$$

$$I_{f_i^{an}} = (I_{f_i})^{an} \qquad \hat{O}_{Z^{rig},z} \cong \hat{O}_{Z,z}$$

$$(I_{f_1})^{an} = (I_{f_2})^{an} \Rightarrow I_{f_1} = I_{f_2} \Rightarrow f_1 = f_2$$

Prop. Y rigid K-space, then

$$\{Y \to \mathbb{A}_K^{n,rig}\} \longleftrightarrow O_Y(Y)^n.$$

Pf. $\mathbb{A}_K^{n,rig} \to \mathbb{A}_K^n \rightsquigarrow$ bijection $\mathrm{Hom}_K(Y, \mathbb{A}_K^{n,rig}) \xrightarrow{\sim} \mathrm{Hom}_K(Y, \mathbb{A}_K^n)$

$\overset{\text{Lem}}{\rightsquigarrow} \mathrm{Hom}_K(Y, \mathbb{A}_K^n) \xrightarrow{\sim} \mathrm{Hom}_K(K[X], O_Y(Y)) \xrightarrow{\sim} O_Y(Y)^n$. □

Example (Affine line) Consider $\mathbb{A}_K^{1,rig}$, the rigid ana'n of $\mathbb{A}_K^1$.

$$Sp\, T_1^{(0)} \hookrightarrow Sp\, T_1^{(1)} \hookrightarrow \cdots \hookrightarrow Sp\, T_1^{(n)} \hookrightarrow \cdots$$

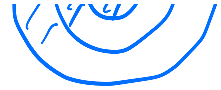
$Sp\, T_1^{(i)}$ is a disk of radius $|c|^i$, centered at origin.

Writing $R^{(i)} = Sp\, T_1^{(i+1)}\langle c^i x^{-1}\rangle$, this is the annulus w/ radii $|c|^i$ and $|c|^{i+1}$, we have

$$Sp\, T_1^{(i+1)} = Sp\, T_1^{(i)} \cup R^{(i)}$$

as an admissible affinoid covering of $Sp\, T_1^{(i+1)}$, hence

$$\mathbb{A}_K^{1,rig} = Sp\, T_1^{(0)} \cup \bigcup_{i \in \mathbb{Z}_{\geq 0}} R^{(i)}$$



is an admissible affinoid covering.

$$\mathbb{A}^{1,\mathrm{rig}}_K - \{0\} = \bigcup_{i\in\mathbb{Z}_{\geq 0}} R^{(i)}.$$

Now assume K is algebraically closed, but not spherically closed. e.g. $\mathbb{C}_p$.

$\exists\, D_0 \supset D_1 \supset \cdots$ of $D^-(a,r)$ in K s.t. $\bigcap_i D_i = \emptyset$. $\quad r \in |K|$

Now $\mathbb{B}^1 = \mathrm{Sp}\, T_1$ as unit disk, WMA all D_i are inside $\mathbb{B}^1$.

Then $\mathbb{B}^1 = \bigcup_i (\mathbb{B}^1 - D_i)$, which is not admissible.

And we have $K = \bigcup_i (\mathbb{B}^1 - D_i) \cup \bigcup_{i\geq 1} R^{(i)}$.