1. Coherent sheaves on rigid spaces.
2. Grothendieck cohomology
3. Proper mapping theorem. (introduction and applications)

1. Consider $X = \mathrm{Sp}A$ and M an A-mod.
   Let $\tilde{M}$ the sheaf extending the functor

$$X' = \mathrm{Sp}A' \subset X \longmapsto M \otimes_A \underset{\sim}{A'} \cdot \mathcal{O}_X(X')$$

We call the sheaf $\tilde{M}$ the $\mathcal{O}_X$-mod associated to M.
written as $M \otimes_A \mathcal{O}_X$

Then $\tilde{M}|_{X'} = M \otimes_A \mathcal{O}_X|_{X'}$.

Prop: The functor $- \otimes_A \mathcal{O}_X$

$$\begin{aligned} A\text{-mod} &\longrightarrow \mathcal{O}_X\text{-mod}. \\ M &\longmapsto M \otimes_A \mathcal{O}_X. \end{aligned}$$

is fully faithful, and commutes with ker, im, tensor, coker.

Further, $0 \to M_1 \to M \to M_2 \to 0$ exact

$\Leftrightarrow$

$0 \to M_1 \otimes \mathcal{O}_X \to M \otimes \mathcal{O}_X \to M_2 \otimes \mathcal{O}_X \to 0.$ exact.

Proof: $\mathrm{Hom}_A(M, M') \xrightarrow[-\otimes\mathcal{O}_X]{} \mathrm{Hom}_{\mathcal{O}_X}(M \otimes \mathcal{O}_X, M' \otimes \mathcal{O}_X) \xrightarrow[H^0]{} \mathrm{Hom}(M, M')$

$0 \to M_1 \to M \to M_2 \to 0$ exact.

$\Rightarrow$ $0 \to M_1 \otimes A' \to M \otimes A' \to M_2 \otimes A' \to 0$. exact. $X' = \mathrm{Sp}A' \subset X = \mathrm{Sp}A$.
$A' \to A$ flat.

Def: X a rigid k-space. $\mathcal{F}$ $\mathcal{O}_X$-mod.
(i) $\mathcal{F}$ is called of finite type if $\exists$ an admissible covering $\underset{\sim}{(X_i)_{i \in I}}$.
with exact sequences

$$\mathcal{O}_X^{S_i}|_{X_i} \to \mathcal{F}|_{X_i} \to 0.$$

(ii) of finite presentation. if further

$$\mathcal{O}_X^{r_i}|_{X_i} \to \mathcal{O}_X^{s_i}|_{X_i} \to \mathcal{F}|_{X_i} \to 0.$$

(iii) coherent. if it is of finite type and if for every admissible open subspace $U \subset X$. the kernel of

$$\mathcal{O}_X^s|_U \to \mathcal{F}|_U \quad \text{is of finite type.}$$

Example: $X = \mathrm{Sp}\,A$. $\mathcal{O}_X^r = A^r \otimes \mathcal{O}_X$. coherent. Notice A is Noetherian.

any $\mathcal{O}_X$-linear: $\mathcal{O}_X^r \longrightarrow \mathcal{O}_X^s \leftrightsquigarrow A^r \to A^s$.

kernel and cokernel are associated to an A-mod. are of finite type.

Remark: An $\mathcal{O}_X$-mod $\mathcal{F}$ on X is coherent iff $\exists$ admissible affinoid covering. $\mathcal{U} = (X_i)_{i\in I}$ s.t. $\mathcal{F}|_{X_i}$ is associated to a finite $\mathcal{O}_{X_i}(X_i)$-mod.

In this case. we call $\mathcal{F}$ is $\mathcal{U}$-coherent.

$\mathcal{F}|_{X_i} = \mathrm{Coker}(\mathcal{O}_X^r|_{X_i} \to \mathcal{O}_X^s|_{X_i})$ associated $(A_i^r \to A_i^s) \otimes \mathcal{O}_{X_i}$ to an A_i-mod.

Theorem (Kiehl): Let $X = \mathrm{Sp}\,A$ be an affinoid k-space. $\mathcal{F}$ an $\mathcal{O}_X$-mod. Then $\mathcal{F}$ is coherent iff $\mathcal{F}$ is associated to a finite A-mod.

Cor: $\mathcal{F}$ is coherent $\Leftrightarrow$ $\exists$ admissible covering $\mathcal{U} = (X_i)$. $\exists$ exact seq's.

$\Leftrightarrow$ $\forall$ admissible covering $\mathcal{U} = (X_i)$. $\exists$ exact seq's.

Proof: "$\Leftarrow$" ✓.

"$\Rightarrow$" Assume $\mathcal{F}$ is $\mathcal{U}$-coherent.

Lemma 1: $H^1(\mathcal{U}, \mathcal{F}) = 0$.

Lemma 2: If $H^1(\mathcal{U}, \mathcal{F}) = 0$ for all $\mathcal{U}$-coherent $\mathcal{O}_X$-mod $\mathcal{F}$. Then any such module is associated to a finite A-mod.

Recall: Lemmas in 4.3.

4. 5. 6. Tate acyclicity.

4: every covering admits rational refinement.

5. 6: Laurent refinement. by $f_1, \dots, f_r$. $X(f_i^{\pm 1}) = \bigcap X(f_i^{\alpha_i})$

By induction. reduce to the case $\mathcal{U} = (\underset{U_1}{\underset{\shortparallel}{X(f)}},\ \underset{U_2}{\underset{\shortparallel}{X(\tfrac{1}{f})}})$

By assumption. $M_1 = \mathcal{F}(U_1)$, $M_2 = \mathcal{F}(U_2)$, $M_{12} = \mathcal{F}(U_1 \cap U_2)$
are finite over $\underline{A\langle f\rangle}$. $\underline{A\langle f^{-1}\rangle}$, and $\underline{A\langle f, f^{-1}\rangle}$ resp.

The alternating cochain $C_a^{\bullet}(\mathcal{U}, \mathcal{F})$ degenerates to.

$$0 \to M_1 \times M_2 \xrightarrow{d^0} M_{12} \to 0.$$

$H^1 = 0 \Leftrightarrow d^0$ is surjective.

Fix an arbitrary residue on A. and consider the Gauß norm
norm
on $A\langle \xi \rangle \xrightarrow{\ } f$. $A\langle \eta \rangle \xrightarrow{\ } f^{-1}$ $A\langle \xi, \eta\rangle$.

and the residue norm from the canonical epimorphisms.

$$\underset{\eta}{A\langle \xi\rangle} \longrightarrow \underset{f^{-1}}{A\langle f\rangle} = A\langle \xi\rangle / (\xi - f)$$

Choose a constant $\beta > 1$. any $g \in A\langle f, f^{-1}\rangle$.

$$g' = \sum c_{\mu\nu} \xi^{\mu} \eta^{\nu} \in A\langle \xi, \eta\rangle \quad \text{s.t. } c_{\mu\nu} \in A.\ |c_{\mu\nu}| \le \beta |g|$$

enlarge if needed $\beta > 1$. s.t $g \in A\langle f, f^{-1}\rangle$. $\exists\ g^{+} \in A\langle f\rangle$. $g^{-} \in A\langle f\rangle$.

$$g = g^{+}|_{U_1 \cap U_2} + g^{-}|_{U_1 \cap U_2}.$$

Choose elements $v_1', \dots, v_m' \in M_1$ $w_1', \dots, w_n' \in M_2$.
generators as $A\langle f\rangle$-mod. $A\langle f^{-1}\rangle$-mod resp.

Since $\mathcal{F}$ is $\mathcal{U}$-coherent. $v_i = v_i'|_{U_1 \cap U_2}$. $w_j = w_j'|_{U_1 \cap U_2}$.
will generate M_{12} as $A\langle f, f^{-1}\rangle$-mod.

maximal norm

$M_1 \otimes A\langle f, f^{-1}\rangle$ $\shortparallel$ coherent.

Consider
$$(A\langle f\rangle)^m \to M_1. \quad (A\langle f^{-1}\rangle)^n \to M_2. \quad (A\langle f, f^{-1}\rangle)^m \twoheadrightarrow M_{12}.$$
$$e_i \mapsto v_i'. \qquad e_j \mapsto w_j' \qquad e_i \mapsto v_i.$$

residue norm.

Claim: $\exists \alpha > 1$, $\forall \varepsilon > 0$, s.t. $\forall u \in M_{12}$.

$\exists u^+ \in M_1$, $u^- \in M_2$. s.t $|u - (u^+|_{U_1 \cap U_2} + u^-|_{U_1 \cap U_2})| < \varepsilon |u|$.

$|u^+| \leq \alpha |u|$. $|u^-| \leq \alpha |u|$.

Proof of Claim: $M_{12} = M_1 \otimes A\langle f, f^{-1}\rangle = M_2 \otimes A\langle f, f^{-1}\rangle$.

$$v_i = \sum_{j=1}^{n} c_{ij} w_j \qquad c_{ij}, d_{jl} \in A\langle f, f^{-1}\rangle.$$

$$w_j = \sum_{l=1} d_{jl} v_l.$$

$A\langle f^{-1}\rangle \subset A\langle f, f^{-1}\rangle$ dense. $\exists$ $c_{ij}' \in A\langle f^{-1}\rangle$.

$f \in A$.

$$\max_{i,j,l} |c_{ij} - c_{ij}'| \, |d_{jl}| \leq \beta^{-2} \varepsilon$$

We verify the claim holds for $\alpha = \beta^2(\max_{ij} |c_{ij}'| + 1)$

Indeed, consider $u \in M_{12}$. Let $u = \sum a_i v_i$, with $a_i \in A\langle f, f^{-1}\rangle$.

As. $|a_i| \leq \beta |u|$. for all i.

thus $a_i = a_i^+|_{U_1 \cap U_2} + a_i^-|_{U_1 \cap U_2}$. $\quad a_i^+ \in A\langle f\rangle$, $a_i^- \in A\langle f^{-1}\rangle$.

$|a_i^\pm| \leq \beta |a_i|$.

Let $u^+ = \sum_i a_i^+ v_i' \in M_1$.

$u^- = \sum_{i,j} a_i^- c_{ij}' w_i' \in M_2$

Then $|u^+| \leq \max |a_i^+| \leq \beta \max |a_i| \leq \beta^2 |u| \leq \alpha |u|$.

$|u^-| \leq \max |a_i^-| \, |c_{ij}'| \leq \max |a_i^-| \max |c_{ij}'|$

$\leq \beta^2 |u| \max |c_{ij}'| \leq \alpha |u|$.

Thus $u = \sum (a_i^+ + a_i^-) u = u^+ + u^- + \underline{\sum (c_{ij} - c_{ij}') a_i^- w_j}$.

has norm $\leq \beta^{-2} \varepsilon \beta^2 |u| = \varepsilon |u|$.

$\square$.

Proof of Lemma 2:

Choose a covering $\mathcal{U} = (U_i)_{i=1,\dots,n}$ of $X = Sp A$ with $U_i = Sp A_i$.

As $\mathcal{F}$ is coherent, $\mathcal{F}|_{U_i}$ is associated to a finite A_i-mod M_i.

For a point $x \in X$. m_x be the maximal ideal in A corres to x.
$m_x \mathcal{O}_X$ the sheaf of ideal.

Further $m_x \mathcal{F}$ is $\mathcal{U}$-coherent.

Consider the exact seq:

$$0 \to m_x\mathcal{F} \to \mathcal{F} \to \mathcal{F}/m_x\mathcal{F} \to 0.$$

If $U' = SpA' \subset X$ affinoid subdomain of X. $U' \subset U_i$ for some i.
then the seq remains exact after restriction to U'.

$$\Rightarrow \quad 0 \to m_x\mathcal{F}(U') \to \mathcal{F}(U') \to \mathcal{F}/m_x\mathcal{F}\,(U') \to H^1 = 0.$$

$$\rightsquigarrow 0 \to C^\bullet(\mathcal{U}, m_x\mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{F}/m_x\mathcal{F}) \to 0$$

is exact

then $0 \to m_x\mathcal{F}(X) \to \mathcal{F}(X) \to \mathcal{F}/m_x\mathcal{F}\,(X) \to 0.$

Claim: restriction

$$\mathcal{F}/m_x\mathcal{F}\,(X) \to \mathcal{F}/m_x\mathcal{F}\,(U_j) \text{ is bijective for all } j$$
s.t. $x \in U_j$.

Notation: $U' = SpA' \subset X$. s.t. $\mathcal{F}|_{U'} = M' \otimes \mathcal{O}_X|_{U'}$
write $U' \cap U_j = Sp A'_j$

Then $\mathcal{F}/m_x|_{U'}$ is associated to $M'/m_x M'$.

and $M'/m_x \to M'/m_x \otimes_A A'_j \xrightarrow{\sim} M'/m_x \otimes_{A'/m_x} A'_j/m_x$.

is bijective for $x \in U_j$. for $A'_j/m_x \simeq A'/m_x$.

Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}/m_x(X) \arrow[r] \arrow[d] & \prod \mathcal{F}/m_x(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "s.\ bij."] & \prod \mathcal{F}/m_x(U_i \cap U_{i'}) \arrow[d, "s.\ bij."] \\
0 \arrow[r] & \mathcal{F}/m_x(U_j) \arrow[r] & \prod \mathcal{F}/m_x(U_i \cap U_j) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathcal{F}/m_x(U_i \cap U_{i'} \cap U_j)
\end{tikzcd}$$

M_i/m_x $\qquad$ M_i/m_x

$\Rightarrow \quad \mathcal{F}/m_x(X) \to \mathcal{F}/m_x(U_j)$ bij. $\qquad \square$

Now consider
$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, two heads] \arrow[d] & \mathcal{F}/m_x(X) \arrow[d, "bij."] \\
M_j = \mathcal{F}(U_j) \arrow[r, two heads] & \mathcal{F}/m_x(U_j) = M_j/m_x
\end{tikzcd}$$

$\Rightarrow$ $\mathcal{F}/m_x(U_j)$ is generated by the image of $\mathcal{F}(X)$

so is $\mathcal{F}(U_j)$ by NAK. locally at $x \in U_j$.

by shrink. U_j. But the sub-mod. generated by $\mathcal{F}(X)$ in $\underbrace{\mathcal{F}(U_j)}_{=M_j}$ must coincide with M_j.

Therefore take $f_1, \dots, f_s \in \mathcal{F}(X)$. s.t. their images generating all of the M_j's.

Induces an epimorphism:

$$\mathcal{O}_X^s \to \mathcal{F} \to 0.$$

the same apply to the kernel.

$$\mathcal{O}_X^r \to \mathcal{O}_X^s \to \mathcal{F} \to 0.$$

$\mathcal{F} = \operatorname{coker}(\mathcal{O}_X^r \to \mathcal{O}_X^s)$ thus associated to an A-mod.

$\square$.

$\varphi: Y \to X$ be a morphism between rigid k-spaces. $\mathcal{F}$ any $\mathcal{O}_Y$-sheaf on Y.
$\mathcal{G}$ $\mathcal{O}_X$- on X

$\varphi_* \mathcal{F}$. on X. $\quad \exists\ \varphi^* \mathcal{G}$ on Y.
direct image. $\qquad$ inverse. image.

$$\varphi_* \mathcal{F}(U) = \mathcal{F}(\varphi^{-1}(U)).$$

$$\mathrm{Hom}_{\mathcal{O}_Y}(\varphi^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \varphi_*\mathcal{F})$$

Example: $f: Y \to \mathrm{Sp}\,k$. $\quad f_*\mathcal{F} = H^0(Y, \mathcal{F})$

2. Grothendieck cohomology. $\quad X$ rigid k-space.

Def: Injectives I in an abelian category $\mathcal{C}$.

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

$\mathrm{Hom}(-, I)$: is exact.

$$0 \to \mathrm{Hom}(\mathcal{E}'', I) \to \mathrm{Hom}(\mathcal{E}, I) \to \mathrm{Hom}(\mathcal{E}', I) \to 0.$$

Prop: $\mathcal{C} = \{\mathcal{O}_X\text{-mods on } X\}$ has enough injectives.

i.e. $\forall\, \mathcal{F} \in \mathrm{Obj}(\mathcal{C})$, $\exists$ injective I. $\mathcal{F} \hookrightarrow I$. $\square$.

Cor: Every object $\mathcal{F} \in \mathcal{C}$ admits an injective resolution.

$$\begin{array}{l} 0 \to \mathcal{F} \\ \quad\;\;\; \downarrow \\ 0 \to I^0 \to I^1 \to \cdots \end{array} \quad \text{all } I^j\text{'s are injectives.}$$

$$\begin{tikzcd} & & & & \mathrm{coker} \arrow[dr] & \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & I^0 \arrow[rr] \arrow[dr] & & I^1 \arrow[u] \arrow[r] & I^2 \; \cdots \\ & & & \mathrm{Coker} \arrow[ur] & & \end{tikzcd}$$

enough injectives.

Let Φ be a left exact functor. $\mathcal{C} \to$ Ab group.

$$R^i\Phi(\mathcal{F}) = H^i(0 \to \Phi I^0 \to \Phi I^1 \to \cdots)$$

is the i-th derived functor of Φ.

$\varphi: Y \to X$ morphism of rigid spaces.

$\mathcal{F} \mapsto \varphi_*\mathcal{F}$ is left exact. then $R^i\varphi_*$ is defined as above.

For $\varphi: Y \to \mathrm{Sp}\,k \quad R^i\varphi_*\mathcal{F} = H^i(Y, \mathcal{F})$

Derived functor induces the long exact sequence from short.

Φ: left exact. $0 \to \mathcal{E}'' \to \mathcal{E} \to \mathcal{E}' \to 0.$

$$\rightsquigarrow \quad 0 \to \Phi\mathcal{E}'' \to \Phi\mathcal{E} \to \Phi\mathcal{E}' \to R^1\Phi\mathcal{E}'' \to R^1\Phi\mathcal{E} \to R^1\Phi\mathcal{E}' \to \cdots$$

$$\to R^i\Phi\mathcal{E}'' \to R^i\Phi\mathcal{E} \to R^i\Phi\mathcal{E}' \to R^{i+1}\Phi\mathcal{E}'' \to \cdots$$

Comparison to Čech cohomology.

Theorem: Let $\mathcal{U}$ be an admissible covering of rigid k-space X. with $\mathcal{F}$ an $\mathcal{O}_X$-mod. Assume $H^q(U, \mathcal{F}) = 0$, $U = \bigcap_{V \in \mathcal{V} \subset \mathcal{U}} V$, $q > 0$.

Then $\check{H}^q(\mathcal{U}, \mathcal{F}) = H^q(X, \mathcal{F})$

Theorem: Let $\mathcal{F}$ an $\mathcal{O}_X$-mod. $\mathfrak{G}$ a system of admissible open subsets.

s.t. (i) $\mathfrak{G}$ is stable under finite intersections.

(ii) Each admissible covering of an admissible open of X admits an admissible refinement consisting elements in $\mathfrak{G}$.

(iii) $\check{H}^p(U, \mathcal{F}) = 0$ for $p > 0$. $U \in \mathfrak{G}$

Then $\check{H}^q(X, \mathcal{F}) = H^q(X, \mathcal{F})$

Cor: $H^q(X, M \otimes \mathcal{O}_X) = 0$ for $q > 0$. $X = \operatorname{Sp} A$. M an A-mod.

The proper mapping theorem:

Def: Let $X \to Y$ with Y affinoid. $U' \subset U \subset X$ open affinoid subspaces.

We call $U' \subset\subset_Y U$. U' relative compact in U. if.

$\exists$ affinoid generators $f_1, \cdots, f_r \in \mathcal{O}_X(U')$ over $\mathcal{O}_Y(Y)$

$$\mathcal{O}_Y(Y)\langle \xi_i \rangle_{i=1,\cdots,r} \longrightarrow \mathcal{O}_X(U')$$
$$\xi_i \longmapsto f_i.$$

s.t. $U \subset \{x \in U' \mid |f_i(x)| < 1\}$.

Or equivalently: $\exists\ \varepsilon \in \sqrt{|k^*|}$. $0<\varepsilon<1$.

$$U \subset U'(\varepsilon^{-1} f_1, \dots, \varepsilon^{-1} f_r).$$

Lemma: X_1, X_2 over Y. affinoids. $U_i \subset X_i$ affinoid subdomain.

(i) $U_1 \subset\subset_Y X_1 \Rightarrow U_1 \times_Y X_2 \subset\subset_Y X_1 \times_Y X_2$.

(ii) $U_i \subset\subset_Y X_i \Rightarrow U_1 \times_Y U_2 \subset\subset_Y X_1 \times_Y X_2$.

In particular: if $X_1, X_2 \subset X$. $X \to Y$ separated.

$$U_1 \cap U_2 \subset\subset_Y X_1 \cap X_2.$$

Sketch: (i) Take $\mathcal{O}_Y(Y)\langle \xi_i \rangle \twoheadrightarrow \mathcal{O}_{X_1}(U_1)$ s.t. $U_1 \subset X_1(\varepsilon^{-1} f_i)_{i=1}^r$

$\xi_i \longmapsto f_i$.

$$U_1 \times_Y X_2 \subset (X_1 \times_Y X_2)(\varepsilon^{-1} f_i)_{i=1}^r$$

Def: Proper morphism:

$\varphi: X \to Y$ is called proper if:

(i) φ is separated.

(ii) $\exists$ an affinoid covering $(Y_i)_{i\in I}$. of Y.

with two admissible affinoid covering $(X_{ij})_{j=1,\dots,n_i}$. $(X_{ij}')_{j=1,\dots,n_i}$ of $\varphi^{-1}(Y_i)$

s.t. $X_{ij} \subset\subset_{Y_i} X_{ij}'$ for all i, j.

Remark: In adic space version $Sp \rightsquigarrow Spa$.

like. $Var \rightsquigarrow Sch$. GAGA-functor.

then the proper morphism can be defined by valuation criterion.

$+ \cdots$

Proper mapping theorem:

Theorem: $\varphi: X \to Y$ be a proper morphism of rigid K-spaces.
(Kiehl)
and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-mod. Then $R^i\varphi_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-mod
for each $i \geq 0$.

Lemma: Assume $Y = \mathrm{Sp}B \supset \mathrm{Sp}B' = Y'$. an affinoid subdomain.

then $H^0(Y', R^q\varphi_*\mathcal{F}) = H^q(\varphi^{-1}(Y'), \mathcal{F}) = H^q(X, \mathcal{F}) \otimes_B B'$

Cor of thm: Affinoids are never proper over K except finite Sp(K-alg).
$\varphi: X \to Y$ proper.

Application: ①.

Then for any closed analytic subset $A \subset X$.
locally on open affinoid parts on X.
A is Zariski closed.

The image $\varphi(A)$ is closed analytic.

② Stein factorization: $\varphi: X \xrightarrow{\text{proper}} Y' \xrightarrow{\text{finite}} Y$.
inducing $\varphi_*\mathcal{O}_X$.
with connected fiber

③ On GAGA-functor:

For K-scheme X. locally of finite type. $\mathcal{F}$ $\mathcal{O}_X$-mod.

$\rightsquigarrow$ X^{rig} with $\mathcal{F}^{rig}$.

Then. $\mathcal{F}^{rig}/X^{rig}$ coherent $\Leftrightarrow$ $\mathcal{F}/X$ coherent.

Theorem: X/k proper: $H^q(X, \mathcal{F}) \cong H^q(X^{rig}, \mathcal{F}^{rig})$ $q \geq 0$

Theorem: rigidification functor is fully faithful for coherent sheaves.

$\mathcal{F} \longmapsto \mathcal{F}^{rig}$.

X/k proper.

Theorem: X/k proper. $\mathcal{F}'$ coherent $\mathcal{O}_{X^{rig}}$-mod.

Then $\underset{\text{up to Isom}}{\exists !}$ coherent $\mathcal{O}_X$-mod $\mathcal{F}$. s.t. $\mathcal{F}^{rig} = \mathcal{F}'$.