

(Westlake Lecture 1)

Basic Height Theory

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So Motivation

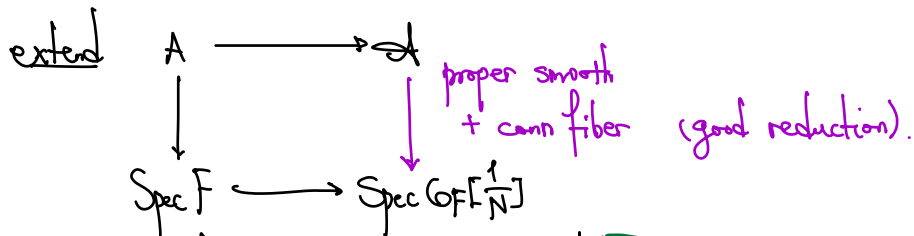
Theorem (Mordell-Weil) Let A be an abelian variety over a number field F .

Then $A(F)$ is a finitely generated abelian group.

Comment: this is also true for function fields.

(Two-step proof)

Step 1 (Weak Mordell-Weil) $\forall n \in \mathbb{N}, \# A(F)/nA(F) < +\infty$.



By "Kummer theory": $A(F)/nA(F) \hookrightarrow H^1(\text{Gal}(F, \mathbb{Q}(\mu_N)), A[n])$
Gal grp of maxl ext'n of F unramified outside Nn .

(from Gal cohom:)

Fact $H^1(G, A[n])$ is finite if it ramifies over finitely many places.

Step 2 Using height theory to understand the "pts" on A .

Eg. $y^2 + y = x^3 - x$, $P = (0, 0) \in E(\mathbb{Q})$.

n	numerator x-coordinate of nP	
2	20	$\approx \log \sim O(n^2)$
4	116	
6	3741	
8	8385	quadratic curve
10	239785	b/c $\hat{h}_L([2]x) = \hat{h}_L([2]x)$
12	59997896	$= \hat{h}_L(x) = 4\hat{h}_L(x)$
14	1849037896	
16	270896443865	
18	16683000076735	

§1 Weil height

Def'n The standard height on $\mathbb{P}^n(\mathbb{Q})$ is $h: \mathbb{P}^n(\mathbb{Q}) \rightarrow \mathbb{R}$

$$h([X_0, \dots, X_n]) := \log \max\{|X_0|, \dots, |X_n|\} \quad (\text{assume } \text{gcd}(X_0, \dots, X_n) = 1).$$

Better Def'n $h(x) = \sum_{v=p \text{ or } \infty} \log \max\{|x_{0v}|, \dots, |x_{nv}|\}$

↑ never need gcd condition.

$$\Leftrightarrow \forall x \in \mathbb{P}^n(F), h(x) = \frac{1}{[F:\mathbb{Q}]} \sum_{v \text{ place of } F} \log \max\{|x_{0v}|, \dots, |x_{nv}|\}.$$

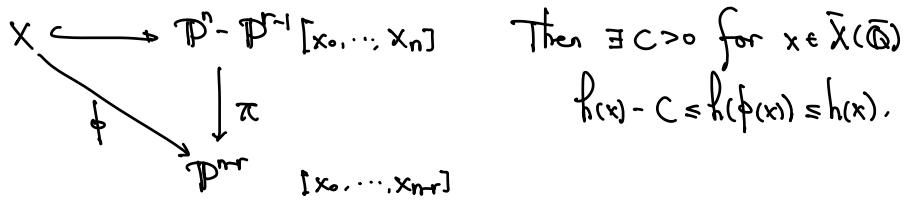
If we have $X \subseteq \mathbb{P}^n$ subvar, then $h|_X: X(\bar{\mathbb{Q}}) \rightarrow \mathbb{R}$.

Basic properties ① For $\phi \in \text{PGL}_{n+1}(\mathbb{Q}) \subset \mathbb{P}^n$, $h(\phi(x)) = h(x) + O(1)$

uniformly bounded by $\sum_{i,j} \log \max\{|\phi_{ij}|, 1\}$.

② X closed subvar in \mathbb{P}^n s.t. $\mathbb{P}^r \cap X = \emptyset$

$$\{[0, \dots, 0, x_{n-r}, \dots, x_n]\}$$



Pf idea: check for $r=0$. \exists homogeneous poly F s.t. $F(0, \dots, 0, 1) \neq 0$

$$\text{s.t. } F=0 \text{ on } X \Rightarrow \forall x \in X, x_n^m + x_n^{m-1}(\dots) + \dots = 0$$

③ For $\phi: \mathbb{P}^n \rightarrow \mathbb{P}^n$ given by homogeneous polys of deg m .

Then $h(\phi(x)) = mh(x) + O(1)$.

Pf: $\mathbb{P}^n \xrightarrow{\phi} \mathbb{P}^{\binom{n+m}{m}-1} \xrightarrow{\text{linear}} \mathbb{P}^n$
 $[x_0, \dots, x_n] \mapsto \text{monomials of deg } m \text{ in } X_i\text{'s}$

the $O(1)$ term comes from this linear rational map.

Theorem (Northcott) Let n, d, M be integers ≥ 1 . Then

$$\#\{x = [x_0, \dots, x_n] \in \mathbb{P}^n(\bar{\mathbb{Q}}) \mid h(x) \leq M, \deg(x) := [\mathbb{Q}(\frac{x_i}{x_j}, i, j=0, \dots, n), \mathbb{Q}] \leq d\} < +\infty$$

Proof. Sketch. Each $[x_0, \dots, x_n]$ can be written uniquely with $x_i \in \mathbb{Z}$, $\text{gcd}(x_0, \dots, x_n) = 1$.

$$h(x) \leq M \Rightarrow \forall i, |x_i| \leq e^M \Rightarrow \text{finite.}$$

When $d > 1$, consider $\underbrace{\mathbb{P}^n \times \dots \times \mathbb{P}^n}_{d \text{ copies}} / S_d$ quotient by S_d -action.
 $\Sigma_{n,d}$

Fact $\Sigma_{n,d} \hookrightarrow \mathbb{P}^N$ projective var, $N = N(n,d)$.

For each $x = [x_0, \dots, x_n] \in \mathbb{P}^n(F)$, $x_i \in F$, $[F:\mathbb{Q}] = d$.

Consider d embeddings $F \hookrightarrow \bar{\mathbb{Q}}$,

$$\text{and } x^\alpha = [\sigma_\alpha(x_0), \dots, \sigma_\alpha(x_n)] \in \mathbb{P}^n(\bar{\mathbb{Q}}), \alpha = 1, \dots, d$$

note: this is a machine
to derive a \mathbb{Q} -pt
from an F -pt.

$$\hookrightarrow (x_1, \dots, x_d) \in (\underbrace{\mathbb{P}^n \times \dots \times \mathbb{P}^n}_{d \text{ copies}}) / S_d(\mathbb{Q})$$

$$\forall \tau \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \tau(x^1, \dots, x^d) = (x^{\tau(1)}, \dots, x^{\tau(d)}) = (x^1, \dots, x^d).$$

$$\Rightarrow \mathbb{P}^n(F) \hookrightarrow \Sigma_{n,d}(\mathbb{Q}). \quad \square$$

§2 Néron-Tate height

The height can be def'd not only on \mathbb{P}^n .

Def'n If L is a very ample line bundle over X/\mathbb{Q} .

$$\text{define } \chi(\bar{\mathbb{Q}}) \xrightarrow{\phi_L} \mathbb{P}^N(\bar{\mathbb{Q}}) = \mathbb{P}(H^0(X, L)^{\otimes d})(\bar{\mathbb{Q}})$$

$$\hookrightarrow h_L(x) := h(\phi_L(x)) \text{ well-def'd up to } O(1).$$

the only ambiguity lies in the choice of basis in the linear system.

Theorem There's a unique map

$$h_L : \text{Pic}(X) \longrightarrow \{\text{Functions on } \chi(\bar{\mathbb{Q}})\} / O(1)$$

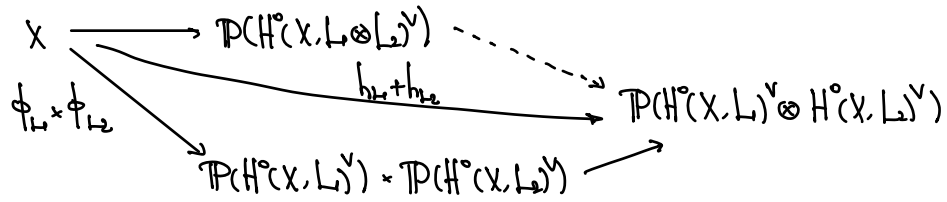
s.t. (1) when L is very ample, h_L is def'd as above.

$$(2) h_{L_1 \otimes L_2} = h_{L_1} + h_{L_2} + O(1). \text{ (homomorphism).}$$

Proof Uniqueness: $\forall L, L = \underbrace{L_1 - L_2}_{L_1 \otimes L_2^{-1}}$ with L_i very ample.

Existence: It suffices to check: if L_1, L_2 very ample, then (2) hold.

$$H^0(X, L_1) \otimes H^0(X, L_2) \longrightarrow H^0(X, L_1 \otimes L_2).$$



Hope: canonical height?

Theorem (Néron-Tate) Let A be an abelian variety / \mathbb{Q} .

There's a unique map

$$\begin{array}{ccc} \text{Pic}(A) & \longrightarrow & \{\text{Functions } A(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}\} \\ L & \longmapsto & \hat{h}_L \end{array}$$

s.t. (1) $\hat{h}_L = h_L + O(1)$

(2) $\hat{h}_{L \otimes L_2} = \hat{h}_L + \hat{h}_{L_2}$

(3) $\psi: B \rightarrow A$ morphism of AV, $\hat{h}_{\psi^*L}(x) = \hat{h}_L(\psi(x))$.

Proof First suppose L is symmetric, i.e. $[-1]^*(L) \cong L$.

Fact: $[m]^*(L) = L^{\otimes m^2}$.

Define: $\hat{h}_L(x) = \lim_{m \rightarrow \infty} \frac{h_L([2^m]_A(x))}{4^m} \in \mathbb{R}$ up to $O(1)$

(Upshot: $|4h_L(x) - h_L([2]_A(x))| < C$ uniformly.)

Similarly, if L is anti-symmetric, $[-1]^*(L) = L^{\otimes -1}$ and $[m]^*(L) = L^{\otimes m}$.

And any line bundle $L = L_{\text{sym}} \otimes L_{\text{anti-sym}}$. □

Prob

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Pic}^0(A) & \rightarrow & \text{Pic}(A) & \rightarrow & \boxed{\text{NS}(A)} \rightarrow 0 \\ & & \oplus & & & & \oplus \\ & & [-1]^*_A = -1 & & & & [-1]^*_A = 1 \end{array}$$

Theorem (1) If L is symmetric, then \hat{h}_L is a quadratic function on $A(\overline{\mathbb{Q}})$ (if L is anti-sym. \hat{h}_L is linear).

⇒ can define a pairing $B_L: A(\bar{\mathbb{Q}}) \times A(\bar{\mathbb{Q}}) \rightarrow \mathbb{R}$

by $B_L(x, y) = \hat{h}_L(x+y) - \hat{h}_L(x) - \hat{h}_L(y)$.

(more canonically $\langle \cdot, \cdot \rangle_{NT}: A(\bar{\mathbb{Q}}) \times A^V(\bar{\mathbb{Q}}) \rightarrow \mathbb{R}$)
 ↑
 Poincaré duality.

(⇒) (Positivity) Moreover, if L is symmetric and ample, $\hat{h}_L(x) \geq 0$
 and $\hat{h}_L(x) = 0 \Leftrightarrow x \in A(\bar{\mathbb{Q}})_{\text{tor}}$.

So $B_L(\cdot, \cdot)$ is a positive-definite symmetric bilinear form on $A(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof (1) Need to show

$$\hat{h}_L(x+y) - \hat{h}_L(x-y) = 2\hat{h}_L(x) + 2\hat{h}_L(y).$$

Theorem of cube $A \times A \times A \xrightarrow{P_1, P_2, P_3} A$

↑
 L line bundle over A .

Then $(p_1 + p_2 + p_3)^* L \otimes (p_1 + p_2)^* L^{\otimes -1} \otimes (p_1 + p_3)^* L^{\otimes -1} \otimes (p_2 + p_3)^* L^{\otimes -1}$

$(x, y, z) \mapsto x+y+z \quad \otimes p_1^* L \otimes p_2^* L \otimes p_3^* L \simeq \text{Triv. bun.}$

Consider $A \times A \rightarrow A \times A \times A$. Eg. $\hat{h}_{(p_1 + p_2)^* L}(x, y) = \hat{h}_L(x+y)$.
 $(x, y) \mapsto (x, y, -y)$

$$\hookrightarrow \hat{h}_L(x+y-y) - \hat{h}_L(x+y) - \hat{h}_L(x-y) - \hat{h}_L(y-y) + \hat{h}_L(x) + \hat{h}_L(y) + \hat{h}_L(-y) = 0.$$

↑ " "
 $\hat{h}_{[-1]^* L}(y) = \hat{h}_L(y)$

(⇒) If $x \in A(\bar{\mathbb{Q}})_{\text{tor}}$, then $\hat{h}_L(x) = \frac{1}{N^2} \hat{h}_L([N]x) = 0$
 (finitely many options)

• if $x \in A(F) \setminus A(F)_{\text{tor}}$, then $\{x, [2]x, [3]x, \dots\} \subseteq A(F)$
 can't all have small height by Northcott.

$\exists N$ s.t. $[N]x \neq 0 \Rightarrow \hat{h}_L(x) = \frac{1}{N^2} \hat{h}_L([N]x) > 0$.

83 Complete proof of Mordell-Weil

Fix $m \geq 2$ symmetric ample line bundle L .

Have proved ① $A(F)/mA(F)$ is finite

② $\forall C > 0, \{x \in A(F) \mid \hat{h}_L(x) < C\}$ finite.

Pick Q_1, \dots, Q_r respectively, $A(F)/2A(F)$.

Take $C = \max\{\hat{h}_L(Q_1), \dots, \hat{h}_L(Q_r)\}$

$\hookrightarrow A(F)^{\hat{h}_L \leq C} = \{P_1, \dots, P_s\}$.

Claim $P_1, \dots, P_s, Q_1, \dots, Q_r$ generate $A(F)$.

Pf. $\forall P \in A(F) \Rightarrow P = 2P' - Q_i$

$$\hat{h}_L(P') = \frac{1}{2}(\hat{h}_L(P - Q_i)) \leq \frac{1}{2}(2\hat{h}_L(P) + 2\hat{h}_L(Q_i)) \leq \hat{h}_L(P) + \frac{1}{2}C. \quad \square$$

~~But~~ This proof doesn't rely on any property of Néron-Tate height.