(Westlake Lecture 1)

Basic Height Theory

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§0 Motivation

Theorem (Mordell-Weil) Let A be an abelian variety over a number field F, Then $A(F)$ is a finitely generated abelian group.

Comment: this is also true for function fields.

(Two-step proof)

Step 1 (Weak Mordell-Weil) $\forall n \in \mathbb{N}$, $\# A(F)/nA(F) < +\infty$.

extend

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\
\operatorname{Spec} F \arrow[r] & \operatorname{Spec} \mathcal{O}_F[\frac{1}{N}]
\end{tikzcd}$$

proper smooth + conn fiber (good reduction).

By "Kummer theory": $A(F)/nA(F) \hookrightarrow H^1(\boxed{\mathrm{Gal}_{F,Nn}}, A[n])$

$\uparrow$ Gal grp of max'l ext'n of F unramified outside Nn.

(from Gal cohom:)

Fact $H^1(G, A[n])$ is finite if it ramifies over finitely many places.

Step 2 Using height theory to understand the "pts" on A.

Eg. $y^2+y = x^3-x$, $P=(0,0) \in E(\mathbb{Q})$.

m	$\lvert$numerator x-coordinate of $mP\rvert$
2	20
4	116
6	3741
8	8385
10	239785
12	59997896
14	1849037896
16	270896443865
18	16683000076735

$\approx \log|\cdot| \sim O(m^2)$.

quadratic curve

b/c $\hat{h}_L([2]x) = \hat{h}_{[2]^*L}(x)$
$= \hat{h}_{4L}(x) = 4\hat{h}_L(x)$.

§1 Weil height

<u>Def'n</u> The <u>standard height</u> on $\mathbb{P}^n(\mathbb{Q})$ is $h : \mathbb{P}^n(\mathbb{Q}) \longrightarrow \mathbb{R}$

$$h([x_0, \cdots, x_n]) := \log \max\{|x_0|, \cdots, |x_n|\} \quad (\text{assume } \gcd(x_0, \cdots, x_n) = 1).$$

<u>Better Def'n</u> $h(x) = \sum_{v = p \text{ or } \infty} \log \max\{|x_0|_v, \cdots, |x_n|_v\}$

$\uparrow$ never need gcd condition.

$\rightsquigarrow \forall x \in \mathbb{P}^n(F)$, $h(x) = \frac{1}{[F:\mathbb{Q}]} \sum_{v \text{ place of } F} \log \max\{|x_0|_v, \cdots, |x_n|_v\}$.

If we have $X \subseteq \mathbb{P}^n$ subvar, then $h|_X : X(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}$.

<u>Basic properties</u> ① For $\phi \in PGL_{n+1}(\mathbb{Q}) \curvearrowright \mathbb{P}^n$, $h(\phi(x)) = h(x) + \boxed{O(1)}$

uniformly bounded by $\sum_w \log \max_{i,j}\{|\phi_{ij}|_w\}$.

② X closed subvar in $\mathbb{P}^n$ s.t. $\mathbb{P}^r \cap X = \phi$

$$\mathbb{P}^r = \{[0, \cdots, 0, x_{n-r}, \cdots, x_n]\}$$

$$\begin{tikzcd} X \arrow[r, hookrightarrow] \arrow[dr, "\phi"'] & \mathbb{P}^n - \mathbb{P}^{r-1} \arrow[d, "\pi"] & {[x_0, \cdots, x_n]} \\ & \mathbb{P}^{n-r} & {[x_0, \cdots, x_{n-r}]} \end{tikzcd}$$

Then $\exists C > 0$ for $x \in \bar{X}(\bar{\mathbb{Q}})$

$$h(x) - C \leq h(\phi(x)) \leq h(x).$$

Pf idea: check for $r = 0$. $\exists$ homogeneous poly F s.t. $F(0, \cdots, 0, 1) \neq 0$

s.t. $F = 0$ on $X \Rightarrow \forall x \in X$, $x_n^m + x_n^{m-1}(\cdots) + \cdots = 0$

③ For $\phi : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ given by homogeneous polys of deg m.

Then, $h(\phi(x)) = m h(x) + O(1)$.

$$\text{Pf}: \begin{tikzcd} \mathbb{P}^n \arrow[r] \arrow[rr, bend left] & \mathbb{P}^{\binom{n+m}{m}-1} \arrow[r, dashed, "\text{linear}"] & \mathbb{P}^N \end{tikzcd}$$

the $O(1)$ term comes from this linear rational map.

$[x_0, \cdots, x_n] \mapsto$ monomials of deg m in X_i's.

<u>Theorem</u> (Northcott) Let n, d, M be integers ≥ 1. Then

$$\#\{x = [x_0, \cdots, x_n] \in \mathbb{P}^n(\bar{\mathbb{Q}}) \mid h(x) \leq M,\ \deg(x) := [\mathbb{Q}(\tfrac{x_i}{x_j}, i,j = 0, \cdots, r), \mathbb{Q}] \leq d\} < +\infty$$

Proof. <u>$d=1$</u>. Each $[x_0, \cdots, x_n]$ can be written uniquely with $x_i \in \mathbb{Z}$, $\gcd(x_0, \cdots, x_n) = 1$.

$$h(x) \leq M \Rightarrow \forall i,\ |x_i| \leq e^M \Rightarrow \text{finite}.$$

When $d > 1$, consider $\underbrace{\mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{d\text{-copies}} / S_d$ quotient by S_d-action.
$$\underbrace{\mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{d\text{-copies}} / S_d \overset{\text{"}}{=} \Sigma_{n,d}.$$

<u>Fact</u> $\Sigma_{n,d} \hookrightarrow \mathbb{P}^N$ projective var, $N = N(n.d)$.

For each $x = [x_0, \cdots, x_n] \in \mathbb{P}^n(F)$, $x_i \in F$. $[F:\mathbb{Q}] = d$.

Consider d embeddings $F \xhookrightarrow{\sigma_i} \bar{\mathbb{Q}}$,

and $x^\alpha = [\sigma_\alpha(x_0), \cdots, \sigma_\alpha(x_n)] \in \mathbb{P}^n(\bar{\mathbb{Q}})$, $\alpha = 1, \cdots, d$

$\rightsquigarrow$ $(x_1, \cdots, x_d) \in \big((\underbrace{\mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{d \text{ copies}})/S_d\big)(\mathbb{Q})$

$\forall \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $\tau(x^1, \cdots, x^d) = (x^{\tau(1)}, \cdots, x^{\tau(d)}) = (x^1, \cdots, x^d)$.

$\Rightarrow \mathbb{P}^n(F) \hookrightarrow \Sigma_{n,d}(\mathbb{Q})$. $\square$

note: this is a machine to derive a $\mathbb{Q}$-pt from an F-pt.

<u>§2 Néron-Tate height</u>

The height can be def'd not only on $\mathbb{P}^n$.

<u>Def'n</u> If L is a very ample line bundle over $X/\mathbb{Q}$,

define $X(\bar{\mathbb{Q}}) \xhookrightarrow{\phi_L} \mathbb{P}^N(\bar{\mathbb{Q}}) = \mathbb{P}(H^0(X, L)^\vee)(\bar{\mathbb{Q}})$

$\rightsquigarrow h_L(x) := h(\phi_L(x))$ well-def'd up to $O(1)$.

the only ambiguity lies in the choice of basis in the linear system.

<u>Theorem</u> There's a unique map

$$h_? : \mathrm{Pic}(X) \longrightarrow \{\text{Functions on } X(\bar{\mathbb{Q}})\}/O(1)$$

s.t. (1) when L is very ample, h_L is def'd as above.

(2) $h_{L_1 \otimes L_2} = h_{L_1} + h_{L_2} + O(1)$. (homomorphism).

<u>Proof</u> Uniqueness: $\forall L$. $L = L_1 - L_2$ with L_i very ample.
$$L_1 - L_2 \overset{\text{"}}{=} L_1 \otimes L_2^{-1}$$

Existence: It suffices to check: if L_1, L_2 very ample, then (2) hold.

$$H^0(X, L_1) \otimes H^0(X, L_2) \longrightarrow H^0(X, L_1 \otimes L_2).$$

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "\phi_{L_1} \times \phi_{L_2}"'] \arrow[rr, bend right=10, "h_{L_1}+h_{L_2}"] & \mathbb{P}(H^0(X, L_1 \otimes L_2)^\vee) \arrow[r, dashed] & \mathbb{P}(H^0(X, L_1)^\vee \otimes H^0(X, L_2)^\vee) \\
& \mathbb{P}(H^0(X, L_1)^\vee) \times \mathbb{P}(H^0(X, L_2)^\vee) \arrow[ur] &
\end{tikzcd}$$

<u>Hope: canonical height?</u>

<u>Theorem</u> (Néron-Tate) Let A be an abelian variety $/\mathbb{Q}$.

There's a unique map

$$\begin{aligned} \mathrm{Pic}(A) &\longrightarrow \{\text{Functions } A(\overline{\mathbb{Q}}) \to \mathbb{R}\} \\ L &\longmapsto \hat{h}_L \end{aligned}$$

s.t. (1) $\hat{h}_L = h_L + O(1)$

(2) $\hat{h}_{L_1 \otimes L_2} = \hat{h}_{L_1} + \hat{h}_{L_2}$

(3) $\psi : B \to A$ morphism of AV, $\hat{h}_{\psi^* L}(x) = \hat{h}_L(\psi(x))$.

<u>Proof</u> First suppose L is <u>symmetric</u>, i.e. $[-1]_A^*(L) \simeq L$.

<u>Fact</u>: $[m]_A^*(L) = L^{\otimes m^2}$.

Define: $\hat{h}_L(x) := \lim_{n \to \infty} \dfrac{h_L([2^n](x))}{4^n} \in \mathbb{R}$ $= h_{L^{\otimes 4^n}}(x) = 4^n h_L(x)$ up to $O(1)$

(Upshot: $|4h_L(x) - h_L([2]_A(x))| < C$ uniformly.)

Similarly, if L is anti-symmetric, $[-1]^*(L) = L^{\otimes -1}$ and $[m]_A^*(L) = L^{\otimes m}$.

And any line bundle $L = L_{sym} - L_{antisym}$. □

<u>Rmk</u>

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0(A) \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & \mathrm{Pic}(A) \arrow[r] & \boxed{NS(A)} \arrow[r] \arrow[d, phantom, "\circlearrowleft"] \arrow[dr, hook, "?"] & 0 \\
& {[-1]_A^* = -1} & & {[-1]_A^* = 1} & \mathbb{Z}
\end{tikzcd}$$

<u>Theorem</u> (1) If L is symmetric, then $\hat{h}_L$ is a quadratic function on $A(\overline{\mathbb{Q}})$

(if L is anti-sym. $\hat{h}_L$ is linear).

$\Rightarrow$ can define a pairing $B_L : A(\bar{\mathbb{Q}}) \times A(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}$

by $B_L(x,y) = \hat{h}_L(x+y) - \hat{h}_L(x) - \hat{h}_L(y)$.

(more canonically $\langle\ ,\ \rangle_{NT} : A(\bar{\mathbb{Q}}) \times \boxed{A^\vee}(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}$)

$\uparrow$ Poincaré duality.

(2) (Positivity) Moreover, if L is symmetric and ample, $\hat{h}_L(x) \geq 0$

and $\hat{h}_L(x) = 0 \Leftrightarrow x \in A(\bar{\mathbb{Q}})_{tor}$.

So $B_L(-,-)$ is a positive-definite symmetric bilinear form on $A(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof (1) Need to show

$$\hat{h}_L(x+y) - \hat{h}_L(x-y) = 2\hat{h}_L(x) + 2\hat{h}_L(y).$$

Theorem of cube $A \times A \times A \xrightarrow{p_1, p_2, p_3} A$

L line bundle over A.

Then $(p_1+p_2+p_3)^* L \otimes (p_1+p_2)^* L^{\otimes -1} \otimes (p_1+p_3)^* L^{\otimes -1} \otimes (p_2+p_3)^* L^{\otimes -1}$

$(x,y,z) \mapsto x+y+z$ $\qquad \otimes p_1^* L \otimes p_2^* L \otimes p_3^* L \simeq$ Triv bun.

Consider $A \times A \longrightarrow A \times A \times A$.
$(x,y) \longmapsto (x,y,-y)$

E.g. $\hat{h}_{(p_1+p_2)^*L}(x,y) = \hat{h}_L(x+y)$.

$$\rightsquigarrow \hat{h}_L(x+y-y) - \hat{h}_L(x+y) - \hat{h}_L(x-y) - \cancel{\hat{h}_L(y-y)} + \hat{h}_L(x) + \hat{h}_L(y) + \underbrace{\hat{h}_L(-y)} = 0.$$

$\parallel$

$\hat{h}_{[-1]^*L}(y) = \hat{h}_L(y)$

(2) If $x \in A(\bar{\mathbb{Q}})_{tor}$, then $\tilde{h}_L(x) = \frac{1}{n^2}\underline{\underline{\hat{h}_L([n]x)}} = 0$

finitely many options

- if $x \in A(F) \setminus A(F)_{tor}$, then $\{x, [2]x, [3]x, \cdots\} \subseteq A(F)$

can't all have small height by Northcott.

$\exists N$ s.t. $[N](x) > 0 \Rightarrow \hat{h}_L(x) = \frac{1}{N^2}\hat{h}_L([N]x) > 0$.

§3 Complete proof of Mordell-Weil

Fix $m \geq 2$ symmetric ample line bundle L.

Have proved ① $A(F)/mA(F)$ is finite

② $\forall C>0$, $\{x \in A(F) \mid \hat{h}_L(x) < C\}$ finite.

Pick $Q_1, \cdots, Q_r$ respectively. $A(F)/2A(F)$.

Take $C = \max\{\hat{h}_L(Q_1), \cdots, \hat{h}_L(Q_r)\}$

$\rightsquigarrow$ $A(F)^{\hat{h}_L \leq C+1} = \{P_1, \cdots, P_s\}$.

Claim $P_1, \cdots, P_s, Q_1, \cdots, Q_r$ generate $A(F)$.

Pf. $\forall P \in A(F) \Rightarrow P = 2P' - Q_i$

$$\hat{h}_L(P') = \frac{1}{4}(\hat{h}_L(P - Q_i)) \leq \frac{1}{4}(2\hat{h}_L(P) + 2\hat{h}_L(Q_i)) \leq \frac{1}{2}\hat{h}_L(P) + \frac{1}{2}C. \quad \square$$

Rmk This proof doesn't rely on any property of Néron-Tate height.