(Westlake Lecture 2)

# Basic Arakelov Theory

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**Tentative Plan** (I) Arakelov divisors and height functions.

(II) Arakelov varieties and intersection numbers.

(III) Ampleness / nefness / bigness of hermitian line bundles.

## §0 Motivation

Function field case: $C$ smooth proj curve $/ k = \bar{k}$, char $k = 0$.

$V$ smooth proj var $/ K = k(C)$ func field of $C$.

$\rightsquigarrow$

$$\begin{array}{ccc} V & & \mathcal{V} \\ \downarrow & & \downarrow \pi \\ K & & C \end{array}$$

$\mathcal{V}$ smooth proj model of $V/K$

generic fibers of $\pi$ are isom to those of $V \to K$.

A rational pt $P \in V(K)$ $\rightsquigarrow$ a rational map $C \to \mathcal{V}$

$\rightsquigarrow$ extended to a morphism $\bar{P} : C \to \mathcal{V}$.

Can do this b/c $C$ sm & $\mathcal{V}$ proj.

- Any divisor $D = \sum n_Y Y$ on $V$ extends to $\bar{D} = \sum n_Y \bar{Y}$ on $\mathcal{V}$

  $\rightsquigarrow$ $\bar{P}^* \bar{D}$ defines a divisor class on $C$

  Caution: the pull-back need not be a divisor but the moving lemma gives a well-defined div class.

  $\rightsquigarrow$ $h_{\bar{D}, \mathcal{V}}(P) := \deg \bar{P}^*(\bar{D})$ for $P \in V(K)$ Weil height function.

**Question** How to transplant this construction to number fields?

(By Arakelov theory).

## §1 Story on number fields

$K$ number field, $V$ sm proj var $/K$.

$$\rightsquigarrow \begin{array}{ccc} V & & \mathcal{V} \\ \downarrow & & \downarrow \pi \\ K & & \mathrm{Spec}\,\mathcal{O}_K \end{array} \quad \text{regular proj model of } V/K.$$

(as $\pi$ is proper)

- A rational pt $P \in V(k)$ extends to a section $\bar{P} : \mathrm{Spec}\,\mathcal{O}_K \to \mathcal{V}$.
- $D$ on $V \rightsquigarrow \bar{D}$ on $\mathcal{V}$ (by taking the Zariski closure)

$\rightsquigarrow \bar{p}^*\bar{D}$ gives a $\underbrace{\text{well-def'd}}$ class on $\mathrm{Spec}\,\mathcal{O}_K$

Comment: Again, it is morally a div when $\mathrm{im}\,\bar{P} \notin \mathrm{Supp}\,\bar{P}^*\bar{D}$.

Say $\bar{p}^*\bar{D} = \sum_{v \in \Sigma_K^f} n_v[v]$, where $\Sigma_K^f = \{\text{finite places on } K\}$

$\Sigma_K^f$ SS closed pts in $\mathrm{Spec}\,\mathcal{O}_K$.

Ambiguity In func-field case: $C$ proj model, complete

But now $\mathrm{Spec}\,\mathcal{O}_K$ is affine & NOT complete.

Question How to define a reasonable degree of $\bar{P}^*(\bar{D})$

(i.e. deg of principal div should be 0)

Let $a \in K^*$ be a rational func on $\mathrm{Spec}\,\mathcal{O}_K$

Then for $\mathrm{div}(a) = \sum_{v \in \Sigma_K^f} \mathrm{ord}_v(a)[v]$,

take $\deg(\mathrm{div}(a)) \overset{?}{=} \sum_{v \in \Sigma_K^f} \mathrm{ord}_v(a) \log N\mathfrak{P}_v$ ($\neq 0$ in general).

the norm of the prime ideal $\leftrightarrow v$.

By Product Formula,

$$\mathrm{div}(a) := \underbrace{\sum_{v \in \Sigma_K^f} \mathrm{ord}_v(a)[v]}_{\text{algebraic Weil div}} - \sum_{v \in \Sigma_K^\infty} \varepsilon_v \log |a|_v [v], \text{ where } \varepsilon_v = \begin{cases} 1, & v \text{ real} \\ 2, & v \text{ complex} \end{cases}$$

**Def'n** An <u>Arakelov divisor</u> on $\operatorname{Spec}\mathcal{O}_K$ is a formal sum

$$D = \sum_{v\in\Sigma_K^f} n_v\cdot[v] + \sum_{v\in\Sigma_K^\infty} \lambda_v\cdot[v].$$

where $n_v\in\mathbb{Z}$, finitely many $n_v\neq 0$, $\lambda_v\in\mathbb{R}$.

<u>Notations</u> $\hat{Z}^1(\operatorname{Spec}\mathcal{O}_K)$ = free abelian grp generated by $D$.

$\hat{R}^1(\operatorname{Spec}\mathcal{O}_K)$ = subgrp of $\hat{Z}^1$ generated by principal Arakelov divs.

$\Rightarrow$ Arakelov div class grp $\boxed{\widehat{CH}^1(\operatorname{Spec}\mathcal{O}_K) := \hat{Z}^1/\hat{R}^1}$.

$\rightsquigarrow$ $\deg : \widehat{CH}^1(\operatorname{Spec}\mathcal{O}_K) \longrightarrow \mathbb{R}$

$$D \longmapsto \sum_{v\in\Sigma_K^f} n_v \log N\mathfrak{P}_v + \sum_{v\in\Sigma_K^\infty}\lambda_v$$

<u>Rmk</u> We have an exact sequence

$$\mathcal{O}_K^\times \xrightarrow{\ell} \mathbb{R}^{r_1+r_2} \longrightarrow \widehat{CH}^1(\operatorname{Spec}\mathcal{O}_K) \longrightarrow Cl(\mathcal{O}_K) \longrightarrow 0$$

regulator

forget the $\infty$-places

$\ell(a) = (\varepsilon_v \log|v(x)|)_{v\in\Sigma_K^\infty}$

if $\mathcal{O}_K=\mathbb{Z}$, then $\widehat{CH}^1(\mathbb{Z})\cong\mathbb{R}$.

Denote $S=\operatorname{Spec}\mathcal{O}_K$ we consider hermitian vector bundles on $S$.

$\rightsquigarrow$ $S(\mathbb{C}) = \operatorname{Hom}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathcal{O}_K) \cong \operatorname{Hom}(\mathcal{O}_K,\mathbb{C}) = \operatorname{Hom}(K,\mathbb{C})$.

Define $K_\mathbb{C} := K\otimes_\mathbb{Q}\mathbb{C}$ by fixing an embedding $\mathbb{Q}\hookrightarrow\mathbb{C}$.

Then $K_\mathbb{C} \cong \bigoplus_{v\in S(\mathbb{C})}\mathbb{C} \cong \operatorname{Hom}(S(\mathbb{C}),\mathbb{C})$

$a\otimes 1 \leftrightarrow \bigoplus_v v(a) \longleftrightarrow$ func $x$ s.t. $x(v)=v(a)$.

Set $F_\infty$ = generator of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ $\rightsquigarrow$ induced involution of $K_\mathbb{C}$.

$\Rightarrow$ $(F_\infty(x))(v) = \overline{x(\bar{v})}$

<u>Say</u> $E$: vector bundle on $S$ (proj $\mathcal{O}_K$-mod of finite rank).

$\rightsquigarrow$ $E_\mathbb{C} := E\otimes K_\mathbb{C}$ and $F_\infty(a\otimes x) = a\otimes F_\infty x$.

**Def'n** A hermitian metric on $K_{\mathbb{C}}$-mod $E_{\mathbb{C}}$ is a map

$$\langle \cdot , \cdot \rangle_E : E_{\mathbb{C}} \times E_{\mathbb{C}} \longrightarrow K$$

s.t. (1) $\langle , \rangle_E$ is linear for the 1st variable

& conj-linear for the 2nd varieble.

(2) $\overline{\langle x, y\rangle_E} = \langle y, x\rangle_E$

(3) $\langle x, x\rangle_E \geq 0$ and $\langle x, x\rangle_E = 0 \Leftrightarrow x = 0$.

**Def'n** A Hermitian vector bundle $E$ on $S$ is

- a vector bundle $E$ on $S$, together with
- a $F_\infty$-inv Hermitian metric on $E_{\mathbb{C}}$

s.t. $\langle F_\infty x, F_\infty y\rangle_E = F_\infty \langle x, y\rangle_E$.

**Rmk** The direct sum, tensor product, dual, and exterior product of Herm v.b.s are all Herm vbs.

For $\wedge$: $\langle e_1 \wedge \cdots \wedge e_n, e_1' \wedge \cdots \wedge e_n' \rangle_{\wedge E} := \det(\langle e_i, e_j' \rangle_E)$.

Consider the arithematic Picard group

$$\widehat{\mathrm{Pic}}(S) := \{\text{the isom classes of Herm line bundles on } S\}.$$

**Theorem** $\exists$ isomorphism $\hat{c}_1 : \widehat{\mathrm{Pic}}(S) \xrightarrow{\sim} \widehat{CH}^1(S)$ (arithematic ver).

(c.f. an algebraic ver: $c_1 : \mathrm{Pic}(S) \cong CH^1(S)$)

**Proof** Choose $\bar{L} \in \widehat{\mathrm{Pic}}(S)$, $L = \mathfrak{p}_1^{s_1} \cdots \mathfrak{p}_t^{s_t}$.

$\rightsquigarrow$ Weil div $z := \sum_{v \in \Sigma_K} n_v [v]$, $n_v = \begin{cases} -s_i, & \text{if } v = \mathfrak{p}_i \\ 0, & \text{otherwise.} \end{cases}$

for $v \in \Sigma_K^\infty$, $\lambda_v = -\boxed{\varepsilon_v} \log \sqrt{\langle 1_{\mathbb{C}}, 1_{\mathbb{C}} \rangle_L (v)}$, $\varepsilon_v$: (real) 1, 2 (cplx)

Then $\hat{c}_1(L) = [z + \sum_{v \in \Sigma_K^\infty} \lambda_v [v]] \in \widehat{CH}^1(S)$. This construction is invertible. □

**Def'n** For $\bar{L} \in \widehat{\mathrm{Pic}}(S)$, $\widehat{\deg}(\bar{L}) := \widehat{\deg}(\hat{c}_1(\bar{L}))$

And $\widehat{\deg}(\bar{E}) := \widehat{\deg}(\det \bar{E})$.

**Rmk** For any global section $s \in L$, define $\mathfrak{a}_s := s \cdot L^{-1} \subseteq \mathcal{O}_K$ (integral ideal)

$$\rightsquigarrow \hat{c}_1(\bar{L}) = \Big[\sum_{v \in \Sigma_K^f} \mathrm{ord}_v(\mathfrak{a}_s)[v] - \sum_{v \in \Sigma_K^\infty} \varepsilon_v \log \sqrt{\langle s, s\rangle_L (v)}\Big].$$

So $\widehat{\deg}(\bar{L}) = \log \# L/s\mathcal{O}_K - \log\Big(\prod_{v \in \mathrm{Hom}(K,\mathbb{C})} |s|_v\Big)$

E.g. $\bar{\mathcal{O}}_K$ trivial bundle with trivial metric,

viewed as a hermitian vector bundle $\bar{E}$ on $\mathrm{Spec}\,\mathbb{Z}$ & $\mathrm{rank} = [K:\mathbb{Q}] = n$

$\rightsquigarrow \mathcal{O}_K = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$

So $\det E = \mathbb{Z}(\overbrace{\omega_1 \wedge \cdots \wedge \omega_n}^{s})$

$\rightsquigarrow |s|_\infty^2 = \langle \omega_1 \wedge \cdots \wedge \omega_n, \omega_1 \wedge \cdots \wedge \omega_n \rangle_{\det E}(\infty)$

$= \det(\langle \omega_i, \omega_j \rangle_{E(\infty)})$

where $\langle \omega_i, \omega_j \rangle_{E(\infty)} = \sum_{v \in \mathrm{Hom}(K,\mathbb{C})} \langle \omega_i, \omega_j \rangle_{\mathcal{O}_K}(v) = \sum_{v \in \mathrm{Hom}(K,\mathbb{C})} v(\omega_i)\overline{v(\omega_j)}$.

$\Rightarrow |s|^2 = \det(A\bar{A}^T)$, $A = (v_i \omega_j)_{i,j}$

$\Rightarrow |s|^2 = |\det A|^2 = |D_K| \Rightarrow \widehat{\deg} E = -\log |D_K|^{1/2}$.

discriminant of $K$.

## Back to the original problem:

**Def'n** $V/K$ sm proj var, $L$ a line bundle on $V$ def'd $/K$. $P \in V(k)$.

Choose a regular proj model $\mathcal{V} \to \mathrm{Spec}\,\mathcal{O}_K$

with $\mathcal{L}$ the model of $L$ on $\mathcal{V}$.

Choose hermitian metric for $\mathcal{L}$, then the <u>metrized height of $P$</u>

is def'd as $h_{\mathcal{V},\bar{\mathcal{L}}}(P) := \frac{1}{[K:\mathbb{Q}]} \widehat{\deg}\, \bar{P}^*(\bar{\mathcal{L}})$.

E.g. $V = \mathbb{P}^n$, $L = \mathcal{O}(1)$, $\mathcal{V} = \mathbb{P}^1_{\mathcal{O}_K}$, $\mathcal{L} = \mathcal{O}_{\mathcal{V}}(1)$.

Let $a_0X_0 + \cdots + a_nX_n$ be a global section to $L$.

The Fubini-Study metric is given by

$$|(a_0X_0 + \cdots + a_nX_n)(P)|^2_{FS} = \left|\left(\frac{(a_0X_0+\cdots+a_nX_n)^2}{X_0^2+\cdots+X_n^2}(P)\right)\right|.$$

$\rightsquigarrow$ $\bar{P}^*(\mathcal{O}(1))$ : $|1_{\mathcal{L}}(P)|^2_{FS} = \dfrac{1}{X_0(P)^2 + \cdots + X_n^2(P)}$.

Exercise. It turns out that

$$h_{\mathcal{V},\mathcal{L}}(P) = \sum_{v \in \Sigma_K^f} \frac{1}{[K:\mathbb{Q}]} \log \max_{0 \le i \le n} |X_i(P)|_v + \sum_{v \in \Sigma_K^\infty} \frac{1}{2[K:\mathbb{Q}]} \varepsilon_v \log\left(\sum_{i=0}^n X_i(P)^2\right)$$

Observation it's differed by some uniformly bounded term from the canonical height.

Prop'n Metrized heights are Weil height functions.

Northcott's Property $\mathcal{L}$ ample.

$\Rightarrow \#\{P \in V(\bar{K}) \mid h_{\mathcal{V},\mathcal{L}}(P) \le B,\ [K(P):K] \le D\} < \infty$ for $B, D \in \mathbb{R}_{>0}$ fixed.

## §2 Arithmetic properties on number fields

to discuss the cycles

$K$ = number field.

Def'n An arithmetic variety is a proj flat scheme over $S = \operatorname{Spec} \mathcal{O}_K$ with sm and geom. irred generic fibres.

note it's not usual to find out such a model for any $V/K$.

$\rightsquigarrow$ $X$ an arith var $/S$.

For any embedding $v : K \hookrightarrow \mathbb{C}$,

$X_v(\mathbb{C})$ is a compact complex manifold.

by projectivity.

Consider $E$ vec bun on $X$ with hermitian metric
being a family of sm metrics on $E_{v,\mathbb{C}}$ on $X_v(\mathbb{C})$ which is $F_\infty$-inv.

**Ⓐ Arakelov intersection pairing** (note the dim'l problem is resolved by relativity (surface = curve over curve))

$X$ regular arith surface $/S$

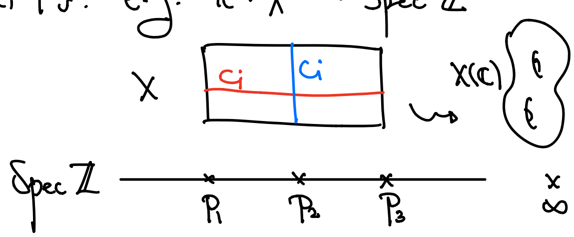
$\rightsquigarrow$ Arakelov divisor $D = D_{fin} + D_{inf}$

$$= \sum_i k_i C_i + \sum_{v \in \Sigma_K^\infty} \lambda_v \cdot F_v$$

Here $\cdot\ \lambda_i \in \mathbb{R},\ k_i \in \mathbb{Z}$

$\cdot\ C_i$ irred closed subscheme of codim 1.

About $F_v$: e.g. $\pi : X \longrightarrow \operatorname{Spec}\mathbb{Z}$




compact Riemann surface.

By properness, there are only two possibilities:

$\pi(C_i) = S$ (horizontal) or $\pi(C_i) = P_i$ (vertical)

We take vertical and irred $C_i$'s as $F_v$.

Upshot This situation differs from the case of curves b/c we have to consider the geometry on the manifold.

For $v \in \Sigma_K^\infty$, choose a hermitian metric on the Riemann surface (at $\infty$).
$\rightsquigarrow X_v(\mathbb{C})$, and denote the assotiated volume form by $d\mu_v$
s.t. $\int_{X_v(\mathbb{C})} d\mu_v = 1$.

**Def'n** The principal Arakelov divisor of a function $f \in K(X)$ is

$$(f) := \underbrace{(f)_{fin}}_{\text{Weil div}} + \sum_{v \in \Sigma_K^\infty} \varepsilon_v \int_{X_v(\mathbb{C})} (-\log|f|_v)\, d\mu_v.$$

Question How to define the intersection number $[D_1, D_2]$ ?

Think of ① $D_2 = F_v \rightsquigarrow$ then define    including when $D_1 = F_v$.

$$[D_1, F_v] = [F_v, D_1] := \begin{cases} 0, & \text{if } D_1 \text{ is vertical} \\ \varepsilon_v m, & \text{if } D_1 \text{ is horizontal and of deg } m \text{ on the generic fiber of } X/S \end{cases}$$

② $D_1, D_2$ distinct, finite irred.

$$[D_1, D_2] = [D_1, D_2]_{fin} + [D_1, D_2]_{inf}$$

where $\cdot$ $[D_1, D_2]_{fin} = \sum \log \#(\mathcal{O}_{X,p}/(f_{1,p}, f_{2,p}))$

$f_{i,p}$ = local equation for $D_i$ at $p$.

$\cdot$ $[D_1, D_2]_{inf} = 0$ if $D_1$ or $D_2$ is a component of a vertical fibre

(For simplicity, $D_1, D_2$ are sections of $p_1, p_2 \in X(K)$.)

$[D_1, D_2]$ should be $\sum_{v \in \Sigma_K^\infty} \varepsilon_v(-\log \underbrace{G_v(p_1, p_2)}_{\text{Green's function}})$

$\bar{L}$ : hermitian line bundle on Riemann surface $X$.

The 1st Chern form $c_1(\bar{L})|_U = \frac{\bar{\partial}\partial}{2\pi i}(-\log|s|^2)$

with $s \in \Gamma(U, L)$, $s|_U \neq 0$.

$\rightsquigarrow \int_X c_1(\bar{L}) = \deg L$

Def'n $\bar{L}$ is admissible if $c_1(\bar{L}) = \deg(L) \cdot d\mu$. (almost being $d\mu$).

Let $Q \in X$, 1 distinguished section of $\mathcal{O}_X(Q)$. $|1|(P) = G(P, Q)$.

Theorem (Arakelov divisor class grp on $X$)

$\cong$

(isom classes of admissible hermitian line bundles on $X$).

## (B) Intersection pairing for general metrics

For $\pi : X \longrightarrow \operatorname{Spec}\mathbb{Z}$,

$$[\bar{L}_1, \bar{L}_2] := \pi_*(\hat{c}_1(\bar{L}_1), \hat{c}_1(\bar{L}_2)).$$

$$= \underbrace{\sum_{x \in X^{(1)}} \log \#(\mathcal{O}_{X,x}/\langle s_1, s_2\rangle)}_{\text{alg part}} - \sum_{\alpha} n_\alpha \log |s_2(p_\alpha)| - \int_{X(\mathbb{C})} \log |s_1| \, c_1(\bar{L}_1)$$

And $\operatorname{div}(s_1|_{X(\mathbb{C})}) = \sum n_\alpha p_\alpha$.

Now $X \longrightarrow \operatorname{Spec} \mathcal{O}_K$ of dim $d+1$,

given $\bar{L}_i$ $(i = 0, \cdots, d)$ hermitian line bundles on $X$.

- $d = 1$ : have def'd by $[\bar{L}_1, \bar{L}_2]$ above.
- General $d (\geq 2)$ : take $s_i \in H^0(X, L_i)$ $(i = 0, \cdots, d)$

  s.t. all $\operatorname{div}(s_i)$ has no common component.

  $\operatorname{div}(s_d) = a_1 Z_1 + \cdots + a_n Z_n$.

$\rightsquigarrow$ $\widehat{\deg}(\bar{L}_0 \bar{L}_1 \cdots \bar{L}_d) = \sum_{i=1}^{n} a_i \widehat{\deg}(\bar{L}_0|_{Z_i} \cdots \bar{L}_{d-1}|_{Z_i})$

$$- \sum_{v \in \Sigma_K^\infty} \int_{X_v(\mathbb{C})} \log |s_d|_v \, c_1(\bar{L}_0) \cdots c_1(\bar{L}_{d-1})$$