(Westlake Lecture 4)

Introduction to the Mordell Conjecture

袁新意.

§1 History

- Poincaré 1901: Chord-tangent Construction.

  "Conjectured": $E/\mathbb{Q}$ elliptic curve $\Rightarrow$ $E(\mathbb{Q})$ fin. gen'd.

- Mordell 1922: proved Poincaré's "conjecture".

  Conjectured if $C/\mathbb{Q}$, proj. smooth curve, genus >1,

  then $C(\mathbb{Q})$ is finite.

- Weil 1928: proved Mordell-Weil thm for Jacobian variety

  over number field J/K ($J(K)$ fin. gen'd).

  there's no abelian variety at that time to replace it.

- Siegel 1935: finiteness of integral points

  $C/\mathbb{Q}$ affine smooth curve, hyperbolic

  ($\text{genus}(\bar{C}) \geq 1$, $C = \bar{C} \setminus \{\text{pts}\}$ affine).

  $\rightsquigarrow$ $C(\mathbb{Z})$ is finite.

  Strategy: Diophantine approximation

- Manin 1963: Mordell conj for function field of curves over $\boxed{\mathbb{C}}$

  not essentially need $\mathbb{C}$

  Strategy: Gauss-Manin connection

- Grauert 1966: Similar result.
- Arakelov 1974: Developed Arakelov Geometry.
- Szpiro 1981: Mordell conj, function field, positive characteristic.
- Faltings 1983: proved Mordell conj.

Mordell conj: K number field. C/K proj. smooth curve, $g>1$.
Then $C(K)$ is finite.

2nd proof Vojta, 1991
3rd proof Lawrence-Venkatesh 2018

§2 Proofs

Faltings' proof

Thm A (Mordell conj)
$\Uparrow$ $C(K)$ finite.

Reason for Thm B $\Rightarrow$ A:
Kodaira-Parshin constr.
$P \in C(K)$ pt. $\rightsquigarrow$ $C_P \longrightarrow C$ finite flat unramified outside P.
$\rightsquigarrow$ $\#\{P\} = \#\{C_P\}$
counting curves.

Thm B (Shafarevich conj. curve).
Fix $g>1$. K, S finite set of places of K.
$\Uparrow$ $\#\{C'/K \text{ curve} \mid g(C')=g,\ C' \text{ good reduction outside } S\} < \infty$

Thm C (Shafarevich conj, abelian variety).
Fix g'. K, S.
$\#\{A/K \mid \dim A = g,\ A \text{ good reduction outside } S\} < \infty$
($C \mapsto \mathrm{Jac}(C)$ "injective" by Torelli's thm).

Main Gredients

① **Thm C** refined by
- $\#\{\text{isogeny classes}\} < \infty$
- fix A_0/K, $\#\{A/K \mid A \text{ isog. to } A_0\} < \infty$.

② isogeny classes $\longrightarrow$ L-function $\leftrightarrow$ Galois rep'ns
$A \longmapsto L(s,A) \longmapsto (G_K \curvearrowright V_\ell(A))$.

③ Faltings Height: $A/K \Rightarrow h(A) \in \mathbb{R}$

Fix a Néron Model: $\mathcal{A} \underset{\pi}{\overset{e}{\rightleftarrows}} \mathrm{Spec}\, \mathcal{O}_K$

ws $h(A) = \widehat{\deg}(e^* \omega_{A/\mathcal{O}_K}, \|\cdot\|_{nat})$

rank $= g$.

$\alpha \in (e^* \Omega^g_{A/\mathcal{O}_K})(\mathbb{C}_\sigma) \cong \Gamma(A_\sigma(\mathbb{C}), \omega_{A_\sigma(\mathbb{C})/\mathbb{C}})$ where $\sigma : K \hookrightarrow \mathbb{C}$.

define $\|\alpha\|^2_{nat} = (-i)^g \int_{A_\sigma(\mathbb{C})} \alpha \wedge \bar{\alpha} \geq 0$.

<u>Thm H</u> (Northcott property). Fix K, g, H. Then

$$\#\{A/K \mid \dim A = g,\ ht(A) < H\} < \infty$$

abelian variety = a pt in moduli space.

proof. $A_g/\mathbb{Q}$ moduli space of abel var, principally polarized. ← A/K AV, $h(A)$

$x = [A] \in A_g(\bar{\mathbb{Q}})$ $h_L(x)$

Weil ht assoc. to Hodge bundle L/A_g.

<u>L-V's proof</u>

<u>Step 1</u> As in Falting's proof, convert to prove

$\pi : X \to Y$ family of curves

(Y/K var, π proj smooth, fibers genus $g > 1$ * non-isotrivial).

Then $Y(K)$ is finite.

<u>Step 2</u> In Falting's proof: (Chebotarev density)

$$\#\{V_p(A)\ G_K\text{-rep} \mid A/K \dim A = g, \text{ good reduction outside } S\} < \infty.$$

<u>Step 3</u> $\exists\, v \mid p$ s.t.

$$\begin{tikzcd} Y(K) \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr] \arrow[dr, bend left] & \{G_K\text{-rep } V_p(J_y) \mid y \in Y(K)\} \arrow[d, "h"] \\ Y(K_v) \arrow[r, "f_v"] & \{G_{K_v}\text{-rep } V_p(J_y) \mid y \in Y(K_v)\} \end{tikzcd}$$

p-adic Hodge map

<u>Thm</u> (LV) f_v is finite-to-one.

So: i inj, f_v finite fibers

$\Rightarrow f_v \circ i = h \circ f$ finite fibers

$\Rightarrow f$ finite fibers

$\Rightarrow Y(k)$ finite by Step 2.