(Westlake lecture 5)

Introduction to Berkovich Spaces

谢俊逸

§1 Banach rings

M abelian grp.

$\rightsquigarrow$ semi-norm $\|\cdot\|: M \to \mathbb{R}_{\geq 0}$ s.t.

(1) $\|0\| = 0$, (2) $\|f-g\| \leq \|f\| + \|g\|$. ← can be strengthen.

It is a norm $\Leftrightarrow$ $\ker \|\cdot\| = \{f \in M, \|f\| = 0\} = \{0\}$.

- Non-archimedean: (2)' $\|f-g\| \leq \max\{\|f\|, \|g\|\}$ $\Rightarrow$ (2).
- A semi-normed space $(M, \|\cdot\|)$ is complete if it is complete as a top space w.r.t. $\|\cdot\|$.

Def'n Comm ring A with complete norm $\|\cdot\|$ s.t.

(1) $\|1\| = 1$, (2) $\|ab\| \leq \|a\| \cdot \|b\|$

if $\|ab\| = \|a\| \cdot \|b\|$, then $\|\cdot\|$ is multiplicative.

$\rightsquigarrow$ A is called a Banach ring.

E.g. (a) $\forall$ ring A. trivial norm $|\cdot|_0$ is

$$|a|_0 = \begin{cases} 1, & a \neq 0 \\ 0, & a = 0 \end{cases}$$ (check: $|\cdot|_0$ is complete).

Punchline Via $|\cdot|_0$: ring A $\rightsquigarrow$ Banach ring A

alg geom $\rightsquigarrow$ analytic geom.

(b) $(\mathbb{Z}, |\cdot|_\infty)$ is a Banach ring.

Complete Euclidean norm

(c) $(\mathbb{R}, |\cdot|)$, $(\mathbb{C}, |\cdot|)$ with archimedean absolute value.

(d) $(\mathbb{Q}_p, |\cdot|_p)$, $(k((t)), \deg)$ non-arch.

(e) Given Banach ring $(A, \|\cdot\|)$, with fixed $r>0$.

Can construct

$$A\{r^{-1}T\} = \{\sum a_i T^i \mid \sum \|a_i\| r^i < \infty\} \quad \text{(arch)}$$

or $$A\{r^{-1}T\} = \{\sum a_i T^i \mid \max \|a_i\| r^i < \infty\} \quad \text{(non-arch)}.$$

$\rightsquigarrow$ $A\{r^{-1}T\}$ is a Banach ring.

$\rightsquigarrow$ $A\{r_1^{-1}T_1, \cdots, r_n^{-1}T_n\}$ $\leftarrow$ important object in Berkovich geom.

(c.f. $k[x_1, \cdots, x_n]$ in AG).

(f) Banach ring $A \supseteq I$ closed ideal.

$g \in A/I \rightsquigarrow \|g\| := \inf_{\bar{f} = g} \|f\|_A$.

note By (e)(f), we have two basic ways to construct many Banach rings.

§2 Spectrum of Banach ring

$(A, \|\cdot\|)$ given, define the spectrum of A by

$$\mathcal{M}(A) := \left\{ \begin{array}{c} |\cdot|_x : A \to [0, \infty) \\ \text{semi-norm} \end{array} \middle| \begin{array}{l} |\cdot|_x \text{ is multiplicative s.t.} \\ |fg|_x = |f|_x |g|_x, \ |f|_x \le \|f\| \end{array} \right\}.$$

bounded.

- topology on $\mathcal{M}(A)$: the weakest top s.t. $\forall f \in A$,

$\ni x \in \mathcal{M}(A) \mapsto |f|_x$ continuous

denoted by $|f(x)|$ for convenient.

- $\phi : A \to B$ bounded morphism

$\rightsquigarrow \phi_* : \mathcal{M}(B) \to \mathcal{M}(A)$.

Theorem. For any Banach A,

(1) $\mathcal{M}(A) \neq \phi$. (2) $\mathcal{M}(A)$ (Hausdorff) compact.

<u>Sketchy proof</u> (2) $\mathcal{M}(A) \hookrightarrow \prod_{f\in A} [0, \|f\|]$ ← product of compact sets

$x \longmapsto (|f(x)|)_{f\in A}$. ← closed image.

(1) <u>Step 1</u> All max'l ideals are closed points in $\mathcal{M}(A)$.

$\Rightarrow$ may assume A = Banach field.

<u>Step 2</u> $\exists$ minimal semi-norm $|\cdot|$ on A $(\leq \|\cdot\|)$.

<u>Step 3</u> $|f^n| = |f|^n$, $\forall n \geq 0$. (Hint: consider $f \mapsto \lim_{n\to\infty} |f^n|^{\frac{1}{n}}$. check: this is a norm.)

Then show that $|f^{-1}| = |f|^{-1}$.

(Assume $|f^{-1}|^{-1} < |f| \Rightarrow \exists$ smaller $|\cdot|'$ s.t. $|f|' = |f^{-1}|^{-1} < |f|$ $\rightsquigarrow$ contradiction.)

<u>Step 4</u> $|f|\cdot|g| \geq |fg| = |f^{-1}g^{-1}|^{-1} \geq (|f^{-1}|\cdot|g^{-1}|)^{-1} = |f|\cdot|g|$. $\square$.

(local)

<u>Examples</u> (0) X = compact top space.

$A = C^0(X, \mathbb{C})$ Banach algebra with $\|\cdot\| = \|\cdot\|_{L^\infty}$.

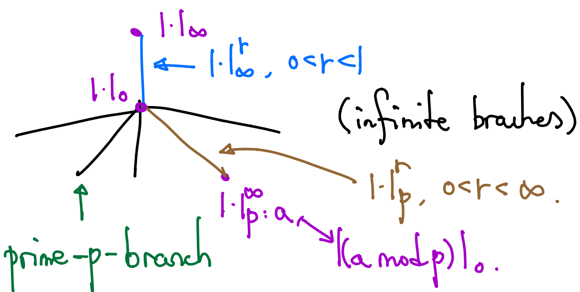
conti functions.

<u>Fact / Thm</u> $\mathcal{M}(A) = X$. (Gelfand - Mazur).

(comparison: $\operatorname{Spec} \Gamma(X, \mathbb{C}) = X$).

(1) k Banach field $\Rightarrow \mathcal{M}(k) = pt$.

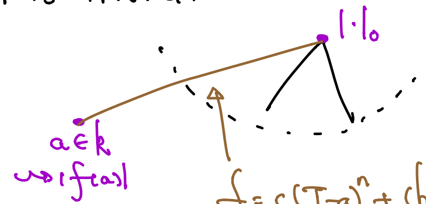
(2) $A = (\mathbb{Z}, |\cdot|_\infty)$. $\mathcal{M}(A)$ looks like



$(k = \bar{k})$

(3) $A = k\{T\}$, with a non-arch valuation $|\cdot|$. $\Rightarrow \mathcal{M}(A) \approx$ disc

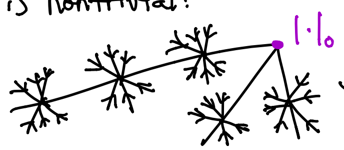
When $|\cdot|$ is trivial:



$f = c(T-a)^n + (\text{higher})$, $f \mapsto e^{-n}$.

$\rightsquigarrow$ $(T-a)$-adic norm.

When $|\cdot|$ is nontrivial:



$\rightsquigarrow$ dense points.

§3 The global story

Set k = either of the following:

(a) field with $|\cdot|_0$; (2) DVF ; (3) $\mathbb{Z}$ with $|\cdot|_\infty$.

X scheme $/k$.

$X = \operatorname{Spec} A$, $X^{an} := \{ |\cdot|_x : A \to \mathbb{R}_{\geq 0}$ multi & s.t. $(|\cdot|_x)|_k \leq |\cdot| \}$ ← in general non-cpt

General case: $X = \bigcup U_i$, $X^{an} = \bigcup U_i^{an}$.

For $x \in X^{an}$, its residue field:

$x \in \operatorname{Spec} A$ for some A $\rightsquigarrow$ $\mathcal{H}(x) := \widehat{\operatorname{Frac}(A/\ker|\cdot|_x)}$.

x induces $\mathcal{M}(\mathcal{H}(x)) \longrightarrow X^{an}$ morphism.

Any morphism $\phi : X \to Y$ $\rightsquigarrow$ $\phi^{an} : X^{an} \to Y^{an}$.

For $y \in Y^{an}$, $X_y^{an} = (\phi^{an})^{-1}(y) / \mathcal{H}(y)$

<u>Properties</u> (1) X separated $\Rightarrow$ X^{an} Hausdorff

(2) X proj. $\Rightarrow$ X^{an} compact

(3) X conn $\Rightarrow$ X^{an} path-conn.

<u>Rmk</u> (1) $K: X^{an} \to X$, "kernel morphism" exists.

(2) $k = \mathbb{Z}$, $|\cdot|$ trivial

$\rightsquigarrow$ $\exists$ natural section of k, $\iota: X \hookrightarrow X^{an}$.

When $X = \operatorname{Spec} A$, $\mathfrak{p} \in A$, $\mathfrak{p} \longmapsto (A \to A/\mathfrak{p})$.

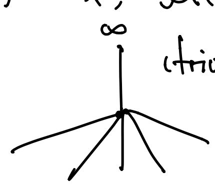
(3) $k = \mathbb{Z}$ or $|\cdot|$ trivial, X/k proj.

$\rightsquigarrow$ $\exists\, r: X^{an} \to X$, $X^{an} = X^{an}[\infty] \sqcup X^{an}[t]$.

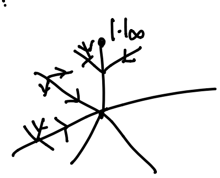
$$\begin{tikzcd}
\operatorname{Spec} \mathcal{H}(x) \arrow[r, "x"] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(\mathcal{H}(x)^\circ) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k
\end{tikzcd}$$

Conn component

E.g. $(\mathbb{P}^1)^{an} = X$, $\mathcal{M}(x)$ looks like:

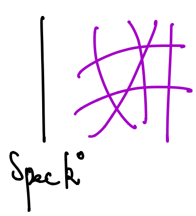


(trivial norm)



(nontrivial norm).

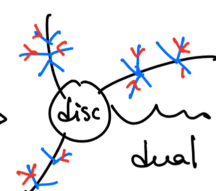
C smooth proj curve.



simple normal crossing

(integral model)

$\rightsquigarrow$



dual object

(finite graph).

<u>§4 Arithmetic divisor</u>

X integral scheme $/k$.

D cartier divisor over X.

support of D

$\rightsquigarrow$ Green's function of D is $g: (X \setminus |D|)^{an} \longrightarrow \mathbb{R}$.

Arithmetic divisor $\bar{D} = (D, g)$

<u>Fact</u> $\bar{D} \geq 0 \iff D \geq 0,\ g \geq 0$

- $\bar{D}$ is called principal $\iff \bar{D} = \widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -2\log|f|)$

alg div   ana div.

- Say $\bar{D}$ or g (norm-)equivariant if

$\forall\, x_1, x_2 \in (X \setminus |D|)^{an}$ s.t. $\underline{|\cdot|_{x_1} = |\cdot|^t_{x_2}}$ for some $t \in \mathbb{R}$,

we have $g(x_1) = t\, g(x_2)$.

Viewed as an action of $\mathbb{R}$ on $\mathcal{U}((X \setminus |D|)^{an})$.

Define $\widehat{\mathrm{Div}}(X^{an})$ = arith divisor grp

$\widehat{\mathrm{Pr}}(X^{an})$ = principal divisor grp $\subseteq \widehat{\mathrm{Div}}(X^{an})_{eqv}$.

$\rightsquigarrow \widehat{\mathrm{CaCl}}(X^{an}) = \widehat{\mathrm{Pic}}(X^{an}) / \widehat{\mathrm{Pr}}(X^{an})$.