(Westlake Lecture 6)

Introduction to the Bogomolov Conjecture

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§1 Statement

$k = \bar{k}$. B = irred proj var $/k$. $\dim B = m \geq 1$.

M = ample line bundle on B.

K = either number field or $k(B)$.

A = abelian var $/K$.

Fix L = symmetric ample line bundle over A.

Note that when K is a number field,

$$\hat{h}(x) = 0 \iff x \text{ is torsion}$$

Caution: Not true for $K = k(B)$.

(Ex $A = A' \otimes_k K$ isotrivial $\rightsquigarrow$ enumorous constant sections

$\rightsquigarrow$ K-pts from k-pts have ht 0.)

essentially the only counterexample.

To describe this explicitly, we introduce

- a trace pair $(A^{\bar{K}/k}, tr)$ = final object of category of (C, f)

  where $C = AV/k$, $f: C \otimes_k \bar{K} \to A_{\bar{K}}$.

  $\rightsquigarrow$ char $k = 0 \Rightarrow (A^{\bar{K}/k}, tr)$ maximal isotrivial subvar of A $/\bar{k}$.

  embedding (essentially)

  char $k = p > 0 \Rightarrow tr$ purely inseparable (not embedding)

  $(A^{\bar{K}/k}, tr)$ "almost" maximal isotrivial.

Prop When $K = k(B)$, $x \in A(\bar{K})$,

$$\hat{h}(x) = 0 \iff x \in \Big(\underbrace{A(\bar{K})_{tor}}_{\substack{\text{torsion part} \\ \text{over } \bar{K}}} + \underbrace{tr\big(A^{\bar{K}/k}(\bar{k})\big)}_{\substack{\text{isotrivial part} \\ \text{over } \bar{k}}}\Big)$$

very large

For an irred subvar $X \subseteq A_{\bar{K}}$, $\varepsilon > 0$,

define $X_\varepsilon := \{ x \in X(\bar{K}) \mid \hat{h}(x) < \varepsilon \}$.

Say X special if $X = tr(Y_{\bar{K}}) + T$, for Y/k, T = torsion coset.

$\rightsquigarrow$ Note if X special, then

(*) $\{ x \in X(\bar{K}) \mid \hat{h}(x) = 0 \}$ is Zariski dense in X

and hence X_ε, $\forall \varepsilon > 0$.

Bogomolov Conjecture (the negative of (*))

If X is not special, then $\exists\, \varepsilon > 0$ s.t. $\overline{X_\varepsilon} \neq X$.

- K = Number field : proved by Ullmo-Zhang (1998)
- Finally proved by Xie-Yuan (2021).

§2 History & Recent process

Special cases (1) K = number field, $C \hookrightarrow Jac(C)$ Bogomolov (1980) } statement.

(2) $K = k(B)$, Yamaki (2013)

(Ingeniously, (2) is more difficult.)

(1) Equidistribution : · Ullmo-Zhang, $K/\mathbb{Q}$.

· Gubler : from non-arch to arch ($\rightsquigarrow k(B)$) (2007)

$K = k(B)$, $\exists$ a totally degenerate place (enormous information).

· Yamaki : Reduce to everywhere-good-reduction. (2006)

$\rightsquigarrow$ Xie-Yuan : Yamaki's thm + Manin-Mumford conjecture (1st technical line).

(2) (2nd tech line) <u>Betti map</u> : char $k=0$.

- Habegger 2013 : $A = E_1 \times E_2$

  (with an exotic idea,

  but the result is weaker than Gubler 2007).

- Gao-Habegger 2019 : $\dim B = 1$
- Cantat-Gao-Habegger-Xie 2021 : char $k=0$, $\dim B > 0$,

  (use some dynamic system + diffe geom.)

  a slightly easier new proof.

(3) (3nd tech line) <u>Admissible metrics</u> : curve $\hookrightarrow$ Jacobian.

(can construct effective versions).

- Zhang 2010 : reduce to graph theory
- Faber 2009 : $g(C) \le 4$.
- Cinkir 2011 : totally proved for $C \hookrightarrow \mathrm{Jac}(C)$

$\Big\}$ char $k=0$

<u>Rmk</u> when char $k=p$, Robin de Jong's trick

to modify the argument in char $k=0$.

<u>Thm</u> (Zhang, Cinkir)

$K = k(B)$, $\dim B = 1$. Pick $\alpha \in \mathrm{Pic}^1(C)$.

Define $i_\alpha : C \hookrightarrow J = \mathrm{Jac}(C)$

$x \longmapsto x - \alpha$.

Néron-Tate height.

$$e(C,\alpha) := \sup_{\substack{F \subseteq C(\bar{K}) \\ \text{finite}}} \ \inf_{x \in C(\bar{K}) \setminus F} h_{NT}(i_\alpha(x))$$

<u>Then</u> if C is not isotrivial,

then $e(C,\alpha) \geq$ const (computable).

depending on g.

Construction of admissible metric

(1) Formal and abstract way.

Setups
- μ : Prob measure on X^{an}.
- $g_\Delta : (C^2)^{an} \setminus \Delta^{an} \to \mathbb{R}$ symmetric Green function.

$\rightsquigarrow$ define a metric $\|\cdot\|_\Delta$ on $\mathcal{O}_{C^2}(\Delta)$ (a line bundle)

$\rightsquigarrow$ $\forall K'/K$, $x \in C(K')$, define $(\mathcal{O}(x), \|\cdot\|_\alpha)$

where $\alpha : \underset{\underset{C_{K'}}{\parallel}}{\operatorname{Spec} K' \times C} \longrightarrow C \times C$.

- $\omega \simeq \Delta^* \mathcal{O}(-\Delta) \rightsquigarrow (\omega, \|\cdot\|_\alpha)$.

Now for any line bundle L on X & metric $\|\cdot\|$ on L,

$\|\cdot\|$ is <u>admissible</u> if $c_1(L, \|\cdot\|) = \deg L \cdot d\mu$.

(2) Graph-theoretic way.