(Westlake Lecture 7)

Uniform Mordell-Lang and uniform Bogomolov (I)

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<u>Setup</u> Curve: proj, smooth, $g>1$.

§1 Mordell conjecture

<u>Recall</u> K number field. C/K curve, $g>1 \Rightarrow |C(K)|<\infty$.

<u>3 Proofs</u> (1) Faltings (defined Faltings height $h(A)$)

$\rightsquigarrow$ counting AVs.

with some p-adic Hodge / moduli space theory involved.

focusing on this

(2) Vojta (Diophantine approximation).

$\rightsquigarrow$ counting Galois rep'ns (via a period map)

Simplified by Faltings, Bomberi.

(3) Lawrence-Venkatesh (p-adic Hodge)

<u>Lang's conj</u> K number field, X/K proj. var, general type (ω_X big).

$X(K)$ not Zariski dense in X. most crucial.

<u>Vojta's proof</u> The idea comes from Mumford's inequality.

<u>Setup</u> Take $\alpha \in \mathrm{Div}(C)$ s.t. $(2g-2)\alpha \overset{\text{lin}}{\sim} \omega_C$.

deg $2g-2$. deg 0 as a divisor

Define the embedding $j: C \hookrightarrow J$, $x \mapsto x-\alpha$.

lin-equiv classes of deg 0 divisors.

Θ-divisor: $\Theta \subseteq J$ of codim 1 ($\dim J = g$)

$\Theta = \{ j(x_1)+\cdots+j(x_{g-1}) \mid x_i \in C \}$.

Θ-divisor is symmetric ample,

s.t. $[m]^*\Theta = m^2\Theta$.

$\rightsquigarrow$ canonical height $\hat{h}_\Theta : J(\bar{k}) \to \mathbb{R}_{\geq 0}$

(quadratic, positive-definite up to torsion)

<u>Notation</u> Denote $\forall x, y \in J(\bar{k})$,

$$|x| = (\hat{h}_\Theta(x))^{1/2}, \quad \langle x, y\rangle = \frac{1}{2}(\hat{h}_\Theta(x+y) - \hat{h}_\Theta(x) - \hat{h}_\Theta(y)).$$

<u>Mumford inequality</u> $\forall x, y \in C(\bar{k})$, $x \neq y$,

$$\underbrace{|x|^2 + |y|^2 - 2g\langle x, y\rangle}_{\text{not positive-def'te}} \geq \underbrace{\boxed{O(1)}}_{\text{not necessarily } >0} \text{ (bounded const).}$$

<u>Idea</u> Poincaré line bundle $P = \underbrace{p_1^*\Theta + p_2^*\Theta - m^*\Theta}$ on $J \times J$.

($m : J \times J \to J$ addition.)

<u>Fact</u> under $j \times j : C \times C \to J \times J$,

$$(j \times j)^* P = p_1^*\alpha + p_2^*\alpha - \Delta \quad \text{in } \mathrm{Pic}(C \times C).$$

e.g. $\quad h_P(x,y) = h_\alpha(x) + h_\alpha(y) - h_\Delta(x,y)$. ("=" up to $O(1)$).

$$h_P(x,y) = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x+y\|^2)$$

$$h_\alpha(x) + h_\alpha(y) = \frac{1}{2g}(|x|^2 + |y|^2)$$

$$h_\Delta(x,y) = \frac{1}{2g}(|x|^2 + |y|^2) - \langle x, y\rangle + O(1).$$

Δ effective $\Rightarrow h_\Delta > O(1)$ on $(C \times C \setminus \Delta)(\bar{k})$.

(Punchline: $x \neq y \Rightarrow (x,y) \notin \Delta$.)

<u>Vojta's idea</u> There are essentially 3 divisors on $C \times C$,

say $p_1^*\alpha$, $p_2^*\alpha$, Δ. $\leftarrow$ Mumford used this.

Define $V = d_1 p_1^*\alpha + d_2 p_2^*\alpha + d\Delta \in \mathrm{Pic}(C \times C)$, $d_1, d_2, d \in \mathbb{Z}$.

$$\rightsquigarrow \quad h_V(x,y) = \frac{d_1}{2g}|x|^2 + \frac{d_2}{2g}|y|^2 - d\langle x, y\rangle + \text{error}$$ $\leftarrow$ deg 2-free.

- Meaningful case

$$g n^2 < (d_1+d)(d_2+d) < g^2 n^2 \quad , \quad g>1.$$

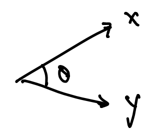
V big, i.e. ht bounded outside each locus

quadratic form $h_V(x,y)$ NOT positive definite.

Vojta inequality Assume $\exists\, c_1, c_2 > 0$ s.t.

$\forall x, y \in C(\bar{k})$, $|x| > c_1$, $|y| > c_2 |x|$.

Then $\langle x, y\rangle < \frac{3}{4}|x|\cdot|y|$. (i.e. "$\cos\theta < \frac{3}{4}$, $\theta > 41°$").



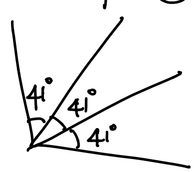
Now, proof of Mordell's conj

$C(k) \hookrightarrow J(k) \simeq \mathbb{Z}^{\oplus r} \oplus \underbrace{\text{tor}}_{\text{unimportant}} \approx \mathbb{Z}^{\oplus r}$ (up to torsion)

$\hookrightarrow \mathbb{R}^{\oplus r}$ with $|\cdot|$ from $\text{ht}^{1/2}$.

Cover $\mathbb{R}^r$ by finitely many cones of angle $41°$.

e.g. $\mathbb{R}^2$: by 9 cones,



} circular cone.

$\rightsquigarrow$ in each cone, Vojta's inequality fails to be valid

$\Rightarrow \forall\, x, y$, $|x| < c_1$ or $|y| < c_2 |x|$.

$\Rightarrow$ $|x|$ is bounded above, $\forall x$.

$\Rightarrow$ only finitely many $x \in C(k)$. $\square$

Comments Vojta's proof gives upper bound of # "big points". $(|x| \gg 0)$.

$\rightsquigarrow$ Seek $\sup\{\hat{h}(x) \mid x \in C(k)\} < ?$

Hopefully (effective Mordell).

$$\sup\{h(x)\} < \boxed{c(g)}\left(h_{\mathrm{Fal}}(J) + \log|d_K|\right).$$

const depending on genus.

- True, over function field by Szpiro.

<u>Punchline</u> "Effective Mordell conj" $\Leftrightarrow$ "abc conj".

(under some appropriate modification).

§2 Uniform Mordell / Bogomolov

<u>Thm</u> (Vojta, Dimitrov-Gao-Habegger, Kühne 2021)

$\exists\ c(g) > 0$ const, depending only on $g \geq 1$

s.t. $|C(K)| \leq c(g)^{1+\mathrm{rk}\, J(K)}$

$\forall$ number field K, curve C/K genus g.

uniform Mordell-Lang, conjectured by Mazur.

<u>Rmk</u> Depending only on C $(g, J(K))$, rather than K (e.g. $K=\mathbb{Q}$).

Stronger: Depending only on g & d_K.

<u>Idea</u> (1) big pts (with large heights):

$$\#\{x \in C(K) \mid \hat{h}(x) > \varepsilon\, h^+_{\mathrm{Fal}}(C)\} < ?$$

(by Vojta, Bombieri, di Diego, Rémond)

(2) small pts

Hopefully $\varepsilon' > \varepsilon$ (but not necessarily).

$\Updownarrow$

essentially equivalent to assume $\varepsilon' = \varepsilon$.

$$\#\{x \in \underbrace{C(\bar{K})} \mid \hat{h}(x) < \varepsilon' h^+_{\mathrm{Fal}}(C)\} < ?$$

suffices to do with K,

but can do with $\bar{K}$. (by DGH & Kühne).

where $h^+_{\mathrm{Fal}}(C) = \max\{h_{\mathrm{Fal}}(C), 1\}$.