(Westlake Lecture 8)

Uniform Mordell-Lang and uniform Bogomolov (II)

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§2 Uniform Mordell/Bogomolov (continued)

Thm (Vojta, Dimitrov-Gao-Habegger, Kühne 2022)

$\exists\ c(g)>0$ const, depending only on $g>1$

s.t. $|C(K)| \le c(g)^{1+\mathrm{rk}\, J(K)}$

$\forall$ number field K, curve C/K genus g.

} uniform Mordell-Lang, conjectured by Mazur.

Rmk Depending only on C $(g, J(K))$, rather than K (e.g. $K=\mathbb{Q}$).

Stronger: Depending only on g & d_K.

Idea (1) big pts (with large heights):

$$\#\{x\in C(K) \mid \hat{h}(x) > \varepsilon h^+_{Fal}(C)\} < ?$$

(by Vojta, Bombieri, di Diego, Rémond)

(2) small pts — Hopefully $\varepsilon' > \varepsilon$ (but not necessarily).

$$\#\{x\in \underbrace{C(\bar{K})}\mid \hat{h}(x) < \varepsilon' h^+_{Fal}(C)\} < ?$$

suffices to do with K, but can do with $\bar{K}$.

$\Updownarrow$ essentially equivalent to assume $\varepsilon'=\varepsilon$.

(by DGH & Kühne).

where $h^+_{Fal}(C) = \max\{h_{Fal}(C), 1\}$.

Bogomolov conj (proved by Ullmo)

C/K, $g>1$. $\exists$ const $\varepsilon>0$ s.t. $\forall \alpha\in \mathrm{Div}(C(\bar{K}))$ with $\deg\alpha=1$,

$$\#\{x\in C(\bar{K}) \mid \hat{h}(\underbrace{x-\alpha}_{\deg=0,\ \text{in } \mathrm{Jac}(C).}) < \varepsilon\} < \infty.$$

$(j_\alpha : C \hookrightarrow J,\ x\mapsto x-\alpha)$ $\quad \varepsilon$ indep of α.

"Uniform": with variant C, K for fixed g ($\rightsquigarrow$ curve family).

Thm (Uniform Bogomolov, DGH, Kühne)

$\exists\, c_1, c_2 > 0$ depending only on $g > 1$,

s.t. $\forall\, C/\bar{\mathbb{Q}}$ of genus g, $\forall\, \alpha \in C(\bar{\mathbb{Q}})$,

$\#\{x \in C(\bar{\mathbb{Q}}) \mid \hat{h}(x-\alpha) < c_1 \cdot h^{+}_{\mathrm{Fal}}(C)\} < c_2$.

Thm (Yuan)

$\exists\, c_1, c_2 > 0$ depending only on $g > 1$,

s.t. $\forall\, K = \mathbb{Q}$ or $\underbrace{K = k(t) \text{ for any field } k}_{\text{function field}}$,

$\forall\, C/K$ of genus g, $\underbrace{\forall\, \alpha \in \mathrm{Pic}^1(C_{\bar{K}})}_{\text{deg 1 divisor.}}$

$\#\{x \in C(\bar{K}) \mid \hat{h}(x-\alpha) < c_1 (h^{+}_{\mathrm{Fal}}(C) + \underbrace{\hat{h}((2g-2)\alpha - \omega_C)}) \} < c_2$.

Proof of [DGH], [K] (only main ingredients)

(1) non-degeneracy result, proved by Ziyang Gao by o-minimality
only work for char 0. (from model theory)

(2) height inequality [DGH]

(3) equidistribution (family version of Ullmo-Zhang's argument) [K].

Proof of Yuan

(1) theory of adelic line bundle of Yuan-Zhang.

(2) bigness of admissible canonical line bundle of family of curves.

§3 Adelic line bundle

Idea Intersection theory over arith/geom quasi-proj var.

$\rightsquigarrow$ limit of line bundles over compactification.

Def'n Over $k = \mathbb{Z}$ or a field

Arakelov geom alg geom.

$\mathcal{U}/k$ quasi-proj, integral, flat

(i) <u>model</u> (compactification)

$\mathcal{X}/k$ proj, int, flat

with $\mathcal{U} \hookrightarrow \mathcal{X}$ open immersion.

(ii) <u>model divisors</u>

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}} = \varinjlim_{\mathcal{X}\text{ model}} \boxed{\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}}$$

k field : $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ usual divisor

$k = \mathbb{Z}$: $\bar{D} = (D, g_D)$, g_D = Green func.

(iii) <u>boundary divisor</u>

Fix model $\mathcal{U} \hookrightarrow \mathcal{X}_0$.

Take D_0 divisor, effective Cartier over $\mathcal{X}_0$

s.t. $|D_0| = \underbrace{\mathcal{X}_0 \setminus \mathcal{U}}$ (support of D_0)

"boundary" $\xrightarrow{\text{blow-up}}$ Cartier div.

$$\text{Take} \quad \bar{D}_0 = \begin{cases} D_0, & \text{if } k \text{ field} \\ (D_0, g_0), & \text{if } k = \mathbb{Z} \end{cases}$$

with $g_0 : \mathcal{X}(\mathbb{C}) \setminus D(\mathbb{C}) \to \mathbb{R}$, $g_0 > 0$

like $-\log|\cdot|$ on boundary.

(iv) $\forall\, \bar{D} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$,

$$\|\bar{D}\|_{\bar{D}_0} = \inf\{a \in \mathbb{Q}_{\boxed{>0}} \mid a\bar{D}_0 \pm \bar{D} \geq 0\}.$$

(<u>Convention</u> $\inf \phi = \infty$).

$\rightsquigarrow \|\cdot\|_{\bar{D}_0} : \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}} \to [0,\infty]$ extended norm.

satisfying triangle inequality.

$\Rightarrow$ Get boundary topology. (indep of choice of $(\mathcal{X}_0, \bar{D}_0)$).

(v) Define $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{Q}} :=$ completion of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$.

called <u>adelic divisors</u> over U/k.

E.g. k field, $\mathcal{X}/k$ proj curve.

$U = \mathcal{X} \setminus P$, $P \in \mathcal{X}$ closed pt.

Choose $(\mathcal{X}_0, \bar{D}_0) = (\mathcal{X}, P)$

$\rightsquigarrow \widehat{\mathrm{Div}}(U)_{\mathrm{mod},\mathbb{Q}} = \mathrm{Div}(\mathcal{X}_0)_{\mathbb{Q}}$.

$\widehat{\mathrm{Div}}(U)_{\mathbb{Q}} = \mathrm{Div}(\mathcal{X}_0)_{\mathbb{Q}} + \mathbb{R}(P) \subseteq \mathrm{Div}(\mathcal{X})_{\mathbb{R}}$.

Rmk $\widehat{\mathrm{Div}}(U)_{\mathbb{Q}} \to \mathrm{Div}(U)$ well-defined.

$(\bar{D}_i) \longmapsto \bar{D}_i|_U$

(vi) Define <u>adelic line bundle</u> over U/k to be a Cauchy sequence $\bar{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \bar{\mathcal{L}}_i), l_i)_{i \geq 1}$.

(a) $\mathcal{L} \in \mathrm{Pic}(U)$

(b) $(\mathcal{X}_i, \bar{\mathcal{L}}_i)$ model of $(U, \mathcal{L})$.

$\bar{\mathcal{L}}_i$: $\mathbb{Q}$-line bundle.

(c) $l_i : \mathcal{L} \to \mathcal{L}_i|_U$ an isomorphism.

(d) $\{\widehat{\mathrm{Div}}(l_i \circ l_1^{-1})\}_i$ is a Cauchy sequence.

$$\begin{tikzcd} \mathcal{L}_1|_U \arrow[r, "l_1^{-1}"] \arrow[rr, bend right] & \mathcal{L} \arrow[r, "l_i"] & \mathcal{L}_i|_U \end{tikzcd}$$

$l_i \circ l_1^{-1}$ rational section of $\mathcal{L}_i \otimes \mathcal{L}_1^{\vee}$.

$\Rightarrow \widehat{\mathrm{Pic}}(U)$ adelic line bundles, U/k quasi-proj.

(vii) K/k function field $\begin{cases} k = \mathbb{Z} : K \text{ number field} \\ k \text{ field} : K/k \text{ function field of 1 variable.} \end{cases}$

X/K quasi-proj.

$\widehat{\mathrm{Pic}}(X) = \varinjlim_{X/K \hookrightarrow U/k} \widehat{\mathrm{Pic}}(U)$ (easy: simply take unions)

Rmk If X/K proj. it will be Zhang's thm.

Properties (1) Absolute intersection U/K

$$\widehat{\mathrm{Pic}}(U)_{\mathrm{nef}}^{\dim U} \longrightarrow \mathbb{R}.$$

(2) (Relative intersection)

Deligne pairing: Suppose $f: U \to V$ proj. flat. of rel dim n

$\rightsquigarrow \widehat{\mathrm{Pic}}(U)_{\mathrm{nef}}^{n+1} \longrightarrow \widehat{\mathrm{Pic}}(V)$.

Def'n (i) $\overline{\mathcal{L}} = ((\mathcal{X}_i, \overline{\mathcal{L}}_i))_{i\geq 1}$ is <u>nef</u>

if $\overline{\mathcal{L}}_i$ nef on $\mathcal{X}_i$, $\forall i \geq 1$.

(ii) $\overline{\mathcal{L}}$ is <u>big</u> if $\mathrm{vol}(\overline{\mathcal{L}}) > 0$.

(if $\overline{\mathcal{L}}$ nef, $\mathrm{vol}(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{\dim U}$, $\overline{\mathcal{L}}$ self-intersection.)

§4 Admissible canonical bundle

$$\begin{array}{c} X \\ \pi \downarrow \\ S \end{array}$$

S/k quasi-proj,

π family of smooth curve of genus $g \geq 1$

(e.g. $S = M_{g,k,N}$).

$\rightsquigarrow \omega_{X/S}$ rel dualizing sheaf.

There is a canonical way to extend $\omega_{X/S} \in \mathrm{Pic}(X)$

to $\overline{\omega}_{X/S} \in \widehat{\mathrm{Pic}}(X)$ with changing the boundary only.

<u>Idea</u> $\overline{L} = j^*\overline{\Theta}$ $\quad \overline{\Theta}$

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "\pi"'] & & J = \mathrm{Jac}(X) \arrow[dl] \\ & S & \end{tikzcd}$$

Θ line bundle over J.

s.t. $[m]^*\Theta = m^2\Theta$.

the most previous motivation of Yuan-Zhang is to make this Θ to be adelic.

$\overset{\text{(if any)}}{\underset{}{+\text{Adelic-ness}}} \Longrightarrow$ $\overline{\Theta} \in \widehat{\mathrm{Pic}}(X)$ extending Θ s.t. $[m]^*\overline{\Theta} = m^2\overline{\Theta}$.

nef

Have $j^*\Theta \approx \omega_{X/S} \Rightarrow j^*\bar{\Theta} \approx \bar{\omega}_{X/S}$ (defining $\bar{\omega}_{X/S}$)

Theorem (Yuan) Fix $X \xrightarrow{\pi} S$.

Assume $S \to M_{g,k}$ generically finite

(natural assumption, to avoid isotrivial part).

Then $\bar{\omega}_{X/S}$ is big (and nef) in $\widehat{\mathrm{Pic}}(X)$.

Idea for uniform Bogomolov:

Why bigness is important?

$\bar{\omega}_{X/S}$ big $\Leftrightarrow \langle \bar{\omega}_{X/S}, \bar{\omega}_{X/S} \rangle$ big by Deligne pairing.

$\Leftrightarrow \langle \bar{L}, \bar{L} \rangle$ big, $\bar{L} = j^*\bar{\Theta}$

"$\Rightarrow$" $h_{\bar{L}} : X(\bar{K}) \to \mathbb{R}$ "big"

i.e. $\#\{x \in X(\bar{K}) \mid h_{\bar{L}}(x) < \varepsilon\}$ small.